EMBEDDING PROBLEMS FOR LIE ALGEBRAS IN ELEMENTARY PARTICLE PHYSICS

Judith M. Ekins

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1973

Embedding Problems for Lie Algebras in Elementary Particle Physics .

A thesis presented by

Judith M. Ekins

to the

University of St Andrews

in application for the

Degree

of

Doctor of Philosophy

September 1973

Declaration

The accompanying thesis is my own composition. It is based upon work carried out by me and no part of it has been previously presented in application for a higher degree.

I was admitted as a research student under Ordinance General No 12 in October 1970 , and as a candidate for the degree under this ordinance in October 1971 .

CERTIFICATE

I certify that the conditions of the Ordinance and Regulations have been fulfilled.

Dr J.F. Cornwell

Research Supervisor

ACKNOWLEDGEMENTS

I should like to express my thanks to the following:

Dr. J.F. Cornwell , my research supervisor , for his constant interest and encouragement, and many suggestions which are unacknowledged in the thesis.

Professor R.B. Dingle, for the facilities made available to me in the department of Theoretical Physics.

The University of St Andrews, which provided financial support throughout the course of this work.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Lie Groups and Lie Algebras in Elementary Particle Physics.

It is well known that many problems may be simplified by examining the symmetry of the system involved , and that the symmetry operations of any system form a group , its symmetry group . Thus although group theory is an elegant mathematical theory in its own right , it is also a very useful tool in several branches of science , especially Physics . Some of its applications are obvious , as for example in crystallography . In other applications , such as in the field of elementary particles , the symmetry properties are less obvious .

The total symmetry group of a system obviously contains , as a subgroup , the symmetry group of the space-time in which the system exists . This subgroup is usually referred to as the external or space-time symmetry group . The choice of this group is a problem in the realm of General Relativity and is by no means solved . Our universe appears to be locally Minkowsky and thus the Poincaré group is an obvious choice , (being the symmetry group of Minkowsky space-time) , but there are other possibilities such as the de Sitter groups .

It is desirable to find the total symmetry group of a system , for then by examining the weights of the irreducible representations , all the equilibrium states, conserved quantities and in some cases the energy spectrum can be predicted . If however the total symmetry group is not known and a subgroup is used instead , then not all possible states will be predicted . This technique is described in many standard texts (e.g. [20] and [38]) and may be applied to classical and quantum mechanical systems .

In the field of elementary particles , the internal symmetries are not well understood , although many equilibrium states and conserved quantities are known . Thus it is a question of finding a group which predicts all known states (and hopefully more) and being able to interpret the symmetry operators physically . In this way it is hoped that the mechanisms involved may be better understood . There are groups like SU(3) which fit some of the physical evidence and predict internal (as opposed to space-time) properties, which might be subgroups of the total symmetry group : such groups are called internal symmetry groups . The problem is then to examine groups containing both the internal and external symmetry groups , coupled in some way . Most internal symmetry groups so far examined (e.g. SU(3) , Sp(6) and the compact form of G_2) are compact Lie groups , while the external symmetry groups are non-compact Lie groups . So any group containing them both will be non-compact. Thus the mathematical problem of embedding Lie groups in non-compact Lie groups arises . A necessary ,(though not sufficient condition), for a Lie group G' to be a subgroup of a Lie group G , is that the Lie algebra $\mathcal{L}'$ of G' is a subalgebra of the Lie algebra $\mathcal{L}$ of G, (see theorem 1, appendix 3) . Thus the problem of embedding Lie algebras is a very important one and leads directly to the embedding of linear Lie groups , as is shown in chapter 4 .

The usual approach to the problem is to consider the Hilbert space H containing all possible equilibrium states of the system as vectors , with all physical processes and measurements of the system characterised as operators , (e.g. the S-matrix is an operator in scattering theory) . These operators will be known as system operators . The total symmetry group is then the group of all operators on H which commute with all the system operators . The system operators are then the Casimir operators of the symmetry group . Any set of generators of the symmetry group correspond to physical quantities, to which quantum numbers can be assigned , e.g. momentum , isospin, strangeness . Some sets of generators are obviously more convenient for a physical interpretation , but these are not often the most convenient mathematically . The embedding problem can best be tackled as a mathematical one and so the physical interpretation of the generators in this case is not obvious .

The simplest method of coupling an internal symmetry group S and an external symmetry group P is in the form of a direct product, $P \otimes S$. This is attractive physically as it implies that the internal and external symmetry operators commute, i.e. space time and internal quantities like charge or hypercharge are in some way intrinsically different , (a doubtful but desirable premise). This is however a dangerous assumption , since this would imply that all particles in a multiplet (set of particles classified by the same irreducible representation) of S would have the same quantum numbers corresponding to the generators and Casimir operators of P (and vice versa). With the Poincaré group this would imply that all particles in the same multiplet have the same mass and spin . With the multiplets of SU(3) and other internal symmetry groups, there is however a sizable mass splitting .(Part but not all of this can be accounted for by assuming that the electro-magnetic symmetry operators do not commute with the Poincaré group.) But by coupling S and P in a different way , it has been suggested that mass splittings might be predicted within the multiplets without having symmetry breaking . Thus the problems of inexact or broken symmetries might be avoided ,and there would be no need to assume an explicit form of symmetry breaking (e.g. Gell-Mann-Okubo mass formula [18]) .

Unfortunately the work of O'Raifeartaigh [34], Jost [25], Segal [35] and others , shows by a series of "<u>No Go</u>" theorems , that coupling the Poincaré group in this way is not easy . Many , e.g. Hegerfeldt and Hennig [23], feel that the only possibilities which do not give a continuous or point mass spectrum are not physically interesting. These possibilities correspond to non-compact large internal symmetry groups. At present we would find difficulty in assigning physical quantities to all the generators , but we must surely allow for the existence of quantum numbers as yet unobserved. Some good results for mass splittings with these larger internal symmetry groups may be possible , as is shown by Flato and Sternheimer [15] with SL(3,C), who by-pass the "No Go" theorems by considering only partially integrable representations [16, 17] . Another possibility is concerned with infinite dimensional fields [40 ,41], where one can more easily obtain formulas for mass splitting, but the physical interpretation is far from obvious .

The study of couplings with other external symmetry groups (e.g. the de Sitter groups [19 , 37] and the conformal group[6]) has made a promising beginning .

It is obvious that a knowledge of the embedding relations of compact and non-compact Lie groups and Lie algebras would be of great help in the search for the symmetry groups of elementary particles and may thus contribute to a better understanding of the subject .

The problem of embedding a compact Lie algebra $\mathcal{L}_c'$ in a compact Lie algebra $\mathcal{L}_c$ is equivalent to embedding $\tilde{\mathcal{L}}'$, the complex extension of $\mathcal{L}_c'$ in $\tilde{\mathcal{L}}$, the complex extension of $\mathcal{L}_c$ [26,27] , and has been studied extensively by Dynkin [14] and Mal'cev [28]. Patera and Sankoff [33] have also listed the maximal subalgebras of compact Lie algebras up to rank 8 , from which it is possible to obtain all subalgebras of compact Lie algebras (of rank 8 or less), by the methods described in section 3.5 .

The technique for embedding a semi-simple Lie algebra $\mathcal{L}'$ in a non-compact semi-simple Lie algebra $\mathcal{L}$ is reviewed in section 1.2 and in chapters 2 and 3 the theory is developed for all the cases which have not been previously examined .

The results for semi-simple Lie algebras can be extended to all Lie algebras ,using Levi's radical splitting theorem and the Mal'cev Harish Chandra theorem (Jacobson [24] pages 91 to 93). Any Lie algebra B can be decomposed into a semi-direct sum $B = \mathcal{L} \oplus C$, where $\mathcal{L}$ is semi-simple and C is the radical of B . All embeddings of $B' = \mathcal{L}' \oplus C'$ in B are conjugate to embeddings where $\mathcal{L}' \subset \mathcal{L}$. Thus before the general problem of embedding one Lie algebra in another can be tackled , the embedding problem for semi-simple Lie algebras must be solved .

Chapter 4 examines the question of embedding one Lie group in another with general results for linear semi-simple Lie groups.

1.2 Embeddings of a real semi-simple Lie algebra $\mathcal{L}'$ in a real semi-simple Lie algebra $\mathcal{L}$.

Cartan [7] and Gantmacher [19] have proved that :
"All different real forms $\mathcal{L}$ of a semi-simple complex Lie algebra $\tilde{\mathcal{L}}$ may be obtained in the following way : first find all involutive automorphisms S of the compact real form $\mathcal{L}_c$ of $\tilde{\mathcal{L}}$. Then take the basis consisting of the 'eigenvectors' of the matrix S , multiplying those eigenvectors having eigenvalue -1 by i and leaving the remaning eigenvectors unchanged . To the basis so obtained there corresponds a real form $\mathcal{L}$ of the given semi-simple Lie algebra $\tilde{\mathcal{L}}$ ". An <u>automorphism</u> is a bi-jection which preserves the Lie product : $[S\underset{\sim}{x}, S\underset{\sim}{y}] = S[\underset{\sim}{x}, \underset{\sim}{y}]$, and <u>involutiveness</u> implies that $S^2 = I$. The <u>trace</u> of all automorphisms S which generate isomorphic real forms is the same and is equal to minus the <u>character</u> δ of the real form $\mathcal{L}$. $\mathcal{L}$ can be denoted as $\sqrt{S}\,\mathcal{L}_c$, where the square root is defined by

$$\sqrt{S} = \frac{1}{2}(1-i)\,S + \frac{1}{2}(1+i)\,I \tag{1.1}$$

Gantmacher [20] has found a set of involutive automorphisms for each simple Lie algebra which generate all the non-isomorphic real forms.

Obviously a necessary condition for $\mathcal{L}' = \sqrt{S'}\,\mathcal{L}'_c$ to be a subalgebra of $\mathcal{L} = \sqrt{S}\,\mathcal{L}_c$ is that $\tilde{\mathcal{L}}'$ is a subalgebra of $\tilde{\mathcal{L}}$, (which implies that $\mathcal{L}'_c$ is a subalgebra of $\mathcal{L}_c$) . In fact $\mathcal{L}'$ may be isomorphic to more than one non-conjugate subalgebra of $\mathcal{L}$.(Two subalgebras are conjugate if each can be mapped isomorphically into the other by a similarity transformation). Any algebra $\mathcal{L}''$ isomorphic to $\mathcal{L}'$ may be considered a faithful representation Γ of $\mathcal{L}'$, $\mathcal{L}'' = \Gamma(\mathcal{L}')$. A representation is a map which preserves the operations of the algebra , in the case of a Lie algebra these are addition and the Lie product (commutation relation), and faithfulness implies the map is one to one (an isomorphism). Thus a necessary condition for $\mathcal{L}'$ to be a subalgebra of $\mathcal{L}$ is that $\Gamma(\mathcal{L}') \subset \mathcal{L}$ for some faithful representation Γ . Cornwell [10] shows that a necessary and sufficient condition for $\mathcal{L}' = \sqrt{S'}\,\mathcal{L}'_c$ to be a subalgebra of $\mathcal{L} = \sqrt{S}\,\mathcal{L}_c$ is that :

there exists an automorphism Y of $\mathcal{L}_c$ such that $Y\underset{\sim}{\Gamma}(\underset{\sim}{b}') = \underset{\sim}{\Gamma}(\underset{\sim}{b}')$ for all $\underset{\sim}{b}' \in \mathcal{L}'_c$ and an automorphism S'_{ext} of $\mathcal{L}_c$, which is an extension of S' ($S'_{ext}\,\underset{\sim}{\Gamma}(\underset{\sim}{b}') = \underset{\sim}{\Gamma}(S'\underset{\sim}{b}')$ for all $\underset{\sim}{b}' \in \mathcal{L}'_c$) such that

$$S = S'_{ext} \circ Y \quad . \tag{1.2}$$

Obviously any embedding representation conjugate to Γ will produce the same results , but all non-conjugate embeddings must be treated separately , as it is possible for $\mathcal{L}'$ to be isomorphic to more than one non-conjugate subalgebra of $\mathcal{L}$. Cornwell [10 , 11, 12] developes this condition (1.2) in the cases where $\tilde{\mathcal{L}}'$ and $\tilde{\mathcal{L}}$ are <u>simple classical</u> Lie algebras . Cases not previously covered are when $\tilde{\mathcal{L}}'$ and/or $\tilde{\mathcal{L}}$ are <u>the direct sum of two (or more) simple Lie algebras</u> and when $\tilde{\mathcal{L}}'$ and/or $\tilde{\mathcal{L}}$ are <u>exceptional Lie algebras</u>. Chapter 2 developes condition (1.2) in the former case and chapter 3 in the latter . Many examples are given especially those concerning embeddings of SL(2,C) , which is the <u>quantum mechanical homogeneous Lorentz group</u> and likely to be a subgroup of any external symmetry group . Some cases considered in chapters 2 and 3 reduce to those considered by Cornwell [10, 11 and 12] and to avoid unnecessary duplication in these cases, the reader is referred to the relevant section of these papers .

1.3 Notations and Definitions.

The summation convention is used throughout unless otherwise stated.

Taking the basis vectors of $\mathcal{L}$, a real Lie algebra to be $\underset{\sim}{x}_i$ ($i = 1,\ldots n$) and the structure constants to be c^k_{ij} ($[\underset{\sim}{x}_i, \underset{\sim}{x}_j] = c^k_{ij}\,\underset{\sim}{x}_k$), an automorphism ad $\underset{\sim}{a}$ of $\mathcal{L}$ can be defined for all $\underset{\sim}{a} \in \mathcal{L}$

$$(\text{ad }\underset{\sim}{a})\,\underset{\sim}{b} = [\underset{\sim}{a}, \underset{\sim}{b}] \qquad (1.3)$$

The **K**illing form B is then defined as

$$B(\underset{\sim}{a}, \underset{\sim}{b}) = \text{trace}(\text{ad }\underset{\sim}{a} \cdot \text{ad }\underset{\sim}{b}) \qquad (1.4)$$

Thus for $\underset{\sim}{a} = a_i\,\underset{\sim}{x}_i$ and $\underset{\sim}{b} = b_i\,\underset{\sim}{x}_i$, $B(\underset{\sim}{a}, \underset{\sim}{b}) = c^q_{jp} c^p_{iq} a_i b_j = \gamma_{ij} a_i b_j$ ($\gamma_{ij} = c^q_{jp}\, c^p_{iq}$). With an appropriate choice of basis (called a canonical form) the γ_{ij} become $\pm\,\delta_{ij}$ and the character of $\mathcal{L}$ is then defined as

$$\delta = \delta_{ij}\,\gamma_{ij} \qquad (1.5)$$

Weyl [39] has shown that every complex Lie algebra $\tilde{\mathcal{L}}$ has one and only one compact real form $\mathcal{L}_c$ up to isomorphism and that for $\mathcal{L}_c$, all the γ_{ij} are negative and hence the character δ of $\mathcal{L}_c$ is $-n$, n being the dimension of $\mathcal{L}_c$).

There are several alternative formulations of the canonical form of $\tilde{\mathcal{L}}$, but that used by Jacobson [24] will be used here. The Cartan subalgebra $\mathcal{H}$ (a maximal set of commuting elements $\underset{\sim}{h}$) has dimension ℓ, where ℓ is the rank of $\tilde{\mathcal{L}}$. Thus for each root α there corresponds a vector $\underset{\sim}{e}_\alpha$ for which $[\underset{\sim}{e}_\alpha, \underset{\sim}{h}] = \alpha(\underset{\sim}{h})\,\underset{\sim}{e}_\alpha$ and a vector $\underset{\sim}{h}_\alpha \in \mathcal{H}$ for which $B(\underset{\sim}{h}_\alpha, \underset{\sim}{h}) = \alpha(\underset{\sim}{h})$. Defining the scaler product (bi-linear form) on the root space as

$$(\alpha, \beta) = B(\underset{\sim}{h}_\alpha, \underset{\sim}{h}_\beta) \qquad (1.6)$$

a basis for $\tilde{\mathcal{L}}$ is then :

$$\left.\begin{aligned} \underset{\sim}{h}_i &= \frac{2\,\underset{\sim}{h}_{\alpha_i}}{(\alpha_i,\alpha_i)} \\ \underset{\sim}{e}_i &= \underset{\sim}{e}_{\alpha_i} \\ \underset{\sim}{f}_i &= \frac{2\,\underset{\sim}{e}_{-\alpha_i}}{(\alpha_i,\alpha_i)} \end{aligned}\right\} \quad \begin{array}{l} i = 1,\ldots.\ \ell \\ \alpha_i \text{ are simple roots} \end{array}$$

and $\underset{\sim}{e}_{i,j,k,\ldots.} = \left[\ldots.. \left[\left[\underset{\sim}{e}_i,\underset{\sim}{e}_j\right],\underset{\sim}{e}_k\right],\ldots\ldots\right]$ and $\underset{\sim}{f}_{i,j,k,\ldots\ldots} = \left[\ldots.. \left[\left[\underset{\sim}{f}_i,\underset{\sim}{f}_j\right],\underset{\sim}{f}_k\right],\ldots\ldots\right.$ for all sets $\{i,j,k,\ldots..\}$ for which $\alpha_i+\alpha_j+\alpha_k+\ldots.$ is a non-simple root.

The following commutation relations hold :

$$\left.\begin{aligned} \left[\underset{\sim}{h}_i,\underset{\sim}{h}_j\right] &= \underset{\sim}{0} \\ \left[\underset{\sim}{e}_i,\underset{\sim}{f}_j\right] &= \delta_{ij}\,\underset{\sim}{h}_j \\ \left[\underset{\sim}{e}_i,\underset{\sim}{h}_j\right] &= A_{ji}\,\underset{\sim}{e}_i \\ \left[\underset{\sim}{f}_i,\underset{\sim}{h}_j\right] &= -A_{ji}\,\underset{\sim}{f}_i \end{aligned}\right\} \quad i,j = 1,\ldots\ldots.\ \ell \qquad (1.7)$$

where the A_{ji} is the (j i)th element of the Cartan matrix for $\tilde{\mathcal{L}}$ (given in appendix 1) . The generators of $\mathcal{L}_c$ are then $i\underset{\sim}{h}_j$ $(j=1,\ldots.\ \ell)$ $\underset{\sim}{e}_k+\underset{\sim}{f}_k$, $i(\underset{\sim}{e}_k-\underset{\sim}{f}_k)$ for $k=1,\ldots\ldots\ \frac{n-\ell}{2}$ (α_k being any root) .

Explicit matrix forms for the defining representations of the simple classical Lie algebras (A_ℓ , B_ℓ , C_ℓ and D_ℓ) in canonical form are well known and given in appendix 2 . A set of explicit matrix forms for the simple exceptional Lie algebras (G_2 ,F_4 , E_6 , E_7 , E_8) are found in section 3.2 .

Every element $\underset{\sim}{g}$ of the compact simply connected group G_c which has $\mathcal{L}_c$ as its Lie algebra provides an automorphism $T(\underset{\sim}{g})$ of $\mathcal{L}_c$, namely $T(\underset{\sim}{g})\,\underset{\sim}{a} = \underset{\sim}{g}\,\underset{\sim}{a}\,\underset{\sim}{g}^{-1}$ for all $\underset{\sim}{a}\in\mathcal{L}_c$. This is known as an inner automorphism, and it can always be expressed in the form $T(\underset{\sim}{u}\ \exp\underset{\sim}{h}\ .\ \underset{\sim}{u}^{-1})$ for some $\underset{\sim}{u}\in G_c$ and some $\underset{\sim}{h}\in\mathcal{H}$. $T(\exp\underset{\sim}{h})$ is infact $\exp(\mathrm{ad}\,\underset{\sim}{h})$ so every inner automorphism can be written as $T(\underset{\sim}{u})\ \exp(\mathrm{ad}\ \underset{\sim}{h})\ T(\underset{\sim}{u})^{-1}$, There may be some automorphisms of $\mathcal{L}_c$ which can not be so expressed ; these are called <u>outer automorphisms</u> and can be written as $T(\underset{\sim}{u}).Z_o.\exp(\mathrm{ad}\ \underset{\sim}{h})T(\underset{\sim}{u})^{-1}$, where $Z_o\underset{\sim}{h} = \underset{\sim}{h}$, and $Z_o.\exp(\mathrm{ad}\ \underset{\sim}{h})$ is a <u>chief outer automorphism</u> ,(see Gantmacher [19]) . It can also be shown that involutive automorphisms of the form $U^{-1}.Z.U$ generate real forms isomorphic to $\sqrt{Z}\ \mathcal{L}_c$. In [19] it is shown that the <u>group of automorphisms</u> $\mathcal{U}$ of $\mathcal{L}_c$ is the sum of <u>connected components</u> $\mathcal{U} = \mathcal{U}_o + \mathcal{U}_1 + \ldots\ldots$, where $\mathcal{U}_o$ is the set of all inner automorphisms and all automorphisms in a component $\mathcal{U}_i$ can be written in the form

$$T(\underset{\sim}{u}).Z_o.\exp(\mathrm{ad}\ \underset{\sim}{h})\ T(\underset{\sim}{u})^{-1} \tag{1.8}$$

with the same Z_o . Each component corresponds to a '<u>particular</u>' <u>rotation</u> of the root space ,(a symmetry operation of the Dynkin diagram). A_ℓ, D_ℓ and E_6 are the only simple Lie algebras with outer automorphisms: D_4 has 6 components ,(only 4 of which contain involutive automorphisms) and the other algebras have 2 components each ($\mathcal{U} = \mathcal{U}_o + \mathcal{U}_1$) . Cornwell [11] has shown that Z_o for A_ℓ can be expressed as

$$Z_o = T(\ \underset{\sim}{d}\ )\ .\ K \tag{1.9}$$

where K is the operation of complex conjugation and the elements of $\underset{\sim}{d}$ are given by

$$d_{jk} = (-1)^{k+1}\ \delta_{k,\ell+2-j} \quad , \quad j,k = 1,\ldots,\ell+1 \tag{1.10}$$

He has also shown that the outer automorphisms of D_ℓ (for $\ell > 2$) can be expressed as $T(\underset{\sim}{q})$, where $\underset{\sim}{q}$ is an orthogonal matrix of the same dimension as the elements of D_ℓ but with determinant -1 ,(see [12]) .

In equation (1.2) the automorphism Y can either be inner or outer . If inner then it can be expressed as $T(\underset{\sim}{c})$, where $\underset{\sim}{c}$ belongs to the <u>centralizer</u> $C(G_c', G_c)$ for the embedding of $\Gamma(G_c')$ in G_c, (i.e. $\underset{\sim}{c}\Gamma(\underset{\sim}{g}') = \Gamma(\underset{\sim}{g}')\ \underset{\sim}{c}$ for all $\underset{\sim}{c} \in C(G_c', G_c)$ and $\underset{\sim}{g}' \in G_c'$).

If Y is outer and $\tilde{\mathcal{L}} = A_\ell$ (or E_6 as shown in chapter 3), then Y is of the form $T(c \circ y) \circ K_{ext}$, where $T(y)\, \Gamma(a') = K_{ext}\, \Gamma(a') = \Gamma(a'^{*})$ for all $a' \in \mathcal{L}_c'$ and for a y in G_c. If $\tilde{\mathcal{L}}$ is D_ℓ then Y is of the form $T(P\, c)$ where $T(P)$ is any outer automorphism for which $T(P)\, \Gamma(a') = \Gamma(a')$ for all $a' \in \mathcal{L}_c'$.
Cornwell [8] shows that when $\tilde{\mathcal{L}}$ is classical and simple all elements c of $C(G_c', G_c)$ can be expressed in the form $r \circ \exp g \circ r^{-1}$, where $r \in C(G_c', G_c)$ and g is a chief centralizer element. For some, but not all, $\mathcal{L}$ it is possible to choose all g to be in $\mathcal{H}$. $P\, c$ may be similarly expressed as $r \circ \exp G \circ r^{-1}$. For a representation $\Gamma = B(\oplus\sum_j I(p_j) \otimes \Gamma^j)\, B^{-1}$, where Γ^j denotes the jth different irreducible representation appearing in the reduction of Γ (Γ^j appears p_j times and has dimension n_j), c is of the form $c = B(\oplus\sum_j c_j')B^{-1} = B(\oplus\sum_j c_j''(p_j) \otimes I(n_j))\, B^{-1}$. c_j' and c_j'' can be expressed in canonical form $c_j' = r_j' \circ \exp g_j' \circ r_j'^{-1}$ and $c_j'' = r_j'' \circ \exp g_j'' \circ r_j''^{-1}$.

Every irreducible representation of a simple complex Lie algebra $\tilde{\mathcal{L}}$ (and hence of $\mathcal{L}_c$ and all other real forms $\mathcal{L}'$ of $\tilde{\mathcal{L}}$) can be labelled either by its dimension n (i.e. Γ_n) or by its highest weight M. M may be expressed in terms of the fundamental weights λ_i of $\tilde{\mathcal{L}}$ $(\lambda_i(h_j) = \delta_{ij}$, $i,j = 1, \ldots, \ell$) : $M = m_i \lambda_i$ or $(m_1, m_2, \ldots, m_\ell)$.

CHAPTER 2

THE CASE WHEN $\tilde{\mathcal{L}}$' AND/OR $\mathcal{L}$ ARE THE DIRECT SUM OF TWO SIMPLE COMPLEX LIE ALGEBRAS

2.1 Introduction

As discussed in chapter 1 , the remaining cases for embedding a real semi-simple Lie algebra $\mathcal{L}'$ in a real non-compact semi-simple Lie algebra $\mathcal{L}$ are when either $\mathcal{L}'$ or $\mathcal{L}$ (or both) are exceptional or not simple .

This chapter is concerned with the case in which at least one ofthe real Lie algebras has a complex extension which is the direct sum * of two simple complex Lie algebras . This theory can easily be extended to the cases in which more than two simple Lie algebras occur in the direct sum . This problem has been discussed briefly by Barut and Raczka [1] for the embedding of SL(2,C) , which is the universal covering group of the homogeneous Lorentz group. By using certain well-known isomorphisms between simple Lie algebras, they obtain by inspection some of the possible embeddings . As will be shown in section 2.9 their list is far from complete .

Section 2.2establishes the form of the Lie algebra of a semi-simple compact Lie group $G_{1c} \otimes G_{2c}$ (G_{1c} and G_{2c} both being simple and compact) ,classifies its automorphisms and examines the real forms generated by its involutive automorphisms. The involutive automorphisms are of two types : those which generate real forms of the type $\mathcal{L}_1 \oplus \mathcal{L}_2$ ($\mathcal{L}_1$ and $\mathcal{L}_2$ being simple) , and those which generate real <u>simple</u> forms . An example of the latter is SL(2,C), which has $A_1 \oplus A_1$ as its complex Lie algebra. The second type of automorphism can be expressed in terms of the automorphism Z , which is defined as $Z(\underset{\sim}{a} \oplus \underset{\sim}{b}) = (\underset{\sim}{b} \oplus \underset{\sim}{a})$, where $\underset{\sim}{a} \oplus \underset{\sim}{b}$ is in $\mathcal{L}_1 \oplus \mathcal{L}_2$ and $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic. The form of Z is examined in detail.

Section 2.3 deals with the embedding of a real simple Lie algebra $\mathcal{L}'$ in real forms of a complex semi-simple Lie algebra $\mathcal{L}_1 \oplus \mathcal{L}_2$, and section 2.4 deals with embedding real forms of a semi-simple complex Lie algebra $\mathcal{L}_1' \oplus \mathcal{L}_2'$ in real forms of another semi-simple complex Lie algebra $\mathcal{L}_1 \oplus \mathcal{L}_2$, of which there are four types . The extension of the automorphism Z and the form of the centralizer are discussed in each case. In sections 2.5 , 2.6 , 2.7 , and 2.8 the four embeddings are examined separately and examples of each given . The embedding discussed in section 2.6 is equivalent to embedding a real form of a semi-simple complex Lie algebra in a real form of a simple complex Lie algebra .

The examples include embeddings of the Lorentz group SL(2,C) and further examples of this are given in section 2.9 .

2.2. The complex Lie algebra of a compact semi-simple Lie group and its real forms

2.2.1 The Lie algebra of $G_{1c} \otimes G_{2c}$ (G_{1c} and G_{2c} being simple and compact).

The real Lie algebra of G_{ic} is denoted by $\mathcal{L}_{ic}$ $(i = 1,2)$, and the real Lie algebra of $G_{1c} \otimes G_{2c}$ by $\mathcal{L}_c$. If the generators of $\mathcal{L}_{ic}$ are $\underset{\sim}{a}_j^{(i)}$ $(j = 1,\ldots m_i)$, then any element $\underset{\sim}{g}_i$ of G_{ic} in a neighbourhood of the identity $\underset{\sim}{I}(N_i)$ can be expressed in the form

$$\underset{\sim}{g}_i = \exp\left(\sum_{j=1}^{m_i} \lambda_j^{(i)}(\underset{\sim}{g}_i)\underset{\sim}{a}_j^{(i)}\right) = \underset{\sim}{I}(N_i) + \sum_{j=1}^{m_i} \lambda_j^{(i)}(\underset{\sim}{g}_i)\underset{\sim}{a}_j^{(i)} + \ldots\ldots .$$

Hence an element $\underset{\sim}{g}_1 \otimes \underset{\sim}{g}_2$ in $G_{1c} \otimes G_{2c}$ in a neighbourhood of the identity $\underset{\sim}{I}(N_1 N_2)$ can be expressed as

$$\underset{\sim}{g}_1 \otimes \underset{\sim}{g}_2 = \underset{\sim}{I}(N_1 N_2) + \sum_{j=1}^{m_1} \lambda_j^{(1)}(\underset{\sim}{g}_1)(\underset{\sim}{a}_j^{(1)} \otimes \underset{\sim}{I}(N_2)) + \sum_{k=1}^{m_2} \lambda_k^{(2)}(\underset{\sim}{g}_2)(\underset{\sim}{I}(N_1) \otimes \underset{\sim}{a}_k^{(2)}) + \ldots .$$

and the generators of the Lie algebra $\mathcal{L}_c$ of $G_{1c} \otimes G_{2c}$, and hence of its complex extension $\tilde{\mathcal{L}}$, are thus given by $\underset{\sim}{x}_j^{(1)} = \underset{\sim}{a}_i^{(1)} \otimes \underset{\sim}{I}(N_2)$ $(j = 1,\ldots m_1)$ and $\underset{\sim}{x}_k^{(2)} = \underset{\sim}{I}(N_1) \otimes \underset{\sim}{a}_k^{(2)}$ $(k = 1,\ldots m_2)$, which obey the commutation relations

$$[\underset{\sim}{x}_j^{(1)}, \underset{\sim}{x}_k^{(2)}] = 0;$$

$$[\underset{\sim}{x}_j^{(1)}, \underset{\sim}{x}_k^{(1)}] = \sum_{\ell=1}^{m_1} c_{jk}^{(1)\ell}\, \underset{\sim}{x}_\ell^{(1)};$$

$$[\underset{\sim}{x}_h^{(2)}, \underset{\sim}{x}_n^{(2)}] = \sum_{\ell=1}^{m_2} c_{hn}^{(2)\ell}\, \underset{\sim}{x}_\ell^{(2)} \quad (j,k = 1,\ldots m_1 \text{ and } h,n = 1,\ldots m_2), \text{ where}$$

$c_{jk}^{(i)\ell}$ are the structure constants of $\tilde{\mathcal{L}}_i$ $(i = 1,2)$. Thus $\tilde{\mathcal{L}}$ is a semi-simple Lie algebra containing two commuting sub-algebras, each of which is an ideal.

There is a Lie algebra which is isomorphic to $\tilde{\mathcal{L}}$ having generators $\hat{\underset{\sim}{x}}_j^{(1)} = \underset{\sim}{a}_j^{(1)} \oplus \underset{\sim}{O}(N_2)$ $(j = 1,\ldots m_1)$ and $\hat{\underset{\sim}{x}}_k^{(2)} = \underset{\sim}{O}(N_1) \oplus \underset{\sim}{a}_k^{(2)}$ $(k = 1,\ldots m_2)$, where $\underset{\sim}{O}(N_i)$ denotes the zero matrix of dimension N_i. Thus the complex Lie algebra of $G_{1c} \otimes G_{2c}$ may be considered in either of the isomorphic forms:

$$\tilde{\mathcal{L}} = (\tilde{\mathcal{L}}_1 \otimes \underset{\sim}{I}(N_2)) + (\underset{\sim}{I}(N_1) \otimes \tilde{\mathcal{L}}_2),$$

$$\tilde{\mathcal{L}} = (\tilde{\mathcal{L}}_1 \oplus \underset{\sim}{O}(N_2)) + (\underset{\sim}{O}(N_1) \oplus \tilde{\mathcal{L}}_2) = \tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2.$$

Since the latter form is more convenient it will be used unless otherwise stated. The real Lie algebra of $G_{1c} \otimes G_{2c}$ will henceforth be denoted by $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$, its complex extension by $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$. A representation $\rho_1 \otimes \rho_2$ of a subgroup of $G_{1c} \otimes G_{2c}$ corresponds to a representation $\rho_1 \oplus \rho_2$ of the corresponding subalgebra of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$.

2.2.2 Automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$.

If T_1 and T_2 are automorphisms of $\mathcal{L}_{1c}$ and $\mathcal{L}_{2c}$ respectively, then they have obvious extensions $\hat{T}_1$ and $\hat{T}_2$, which are automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ namely $\hat{T}_1 (\underset{\sim}{a} \oplus \underset{\sim}{b}) = T_1\underset{\sim}{a} \oplus \underset{\sim}{b}$ and $\hat{T}_2 (\underset{\sim}{a} \oplus \underset{\sim}{b}) = \underset{\sim}{a} \oplus T_2\underset{\sim}{b}$, for all $\underset{\sim}{a} \oplus \underset{\sim}{b}$ in $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$. Automorphisms of the form $\hat{T}_1 \circ \hat{T}_2$ will hereafter be classified as automorphisms of type (i), (the special case of T_1 or T_2 being the identity is included)

If $\mathcal{L}_{1c}$ and $\mathcal{L}_{2c}$ are isomorphic then there is an automorphism Z defined by $Z(\underset{\sim}{a} \oplus \underset{\sim}{b}) = \underset{\sim}{\Phi}^{-1}(\underset{\sim}{b}) \oplus \underset{\sim}{\Phi}(\underset{\sim}{a})$, for all $\underset{\sim}{a}$ in $\mathcal{L}_{1c}$ and $\underset{\sim}{b}$ in $\mathcal{L}_{2c}$, where Φ is the isomorphism $\Phi(\mathcal{L}_{1c}) = \mathcal{L}_{2c}$. Automorphisms of the form $Z \circ \hat{T}_1 \circ \hat{T}_2$ will be called automorphisms of type (ii).

In theorem 2 Appendix 3 it is proved that all automorphisms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$ and hence of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ are of type (i), unless $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ are isomorphic, when they are of type (i) or of type (ii) only.

2.2.3 Involutive automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ and the real forms generated by them.

(a) Automorphisms of type (i)

The involutive condition applied to the automorphism $\hat{T}_1 \circ \hat{T}_2$ implies that both T_1 and T_2 are involutive. The real form generated by $\hat{T}_1 \circ \hat{T}_2$ is $\sqrt{\hat{T}_1 \circ \hat{T}_2} (\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$. $\sqrt{\hat{T}_1 \circ \hat{T}_2}$ is by definition (1.1) given by $\sqrt{\hat{T}_1 \circ \hat{T}_2} = \left\{\frac{1}{2} (1-i)\hat{T}_1 \circ \hat{T}_2 + \frac{1}{2}(1+i) I\right\}$. Thus

$$\sqrt{\hat{T}_1 \circ \hat{T}_2} (\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}) = \left\{\tfrac{1}{2}(1-i)(T_1 \mathcal{L}_{1c} \oplus T_2\mathcal{L}_{2c}) + \tfrac{1}{2}(1+i)(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})\right\}$$

$= \left\{\frac{1}{2}(1-i)T_1 + \frac{1}{2}(1+i)I\right\}\mathcal{L}_{1c} \oplus \left\{\frac{1}{2}(1-i)T_2 + \frac{1}{2}(1+i)I\right\}\mathcal{L}_{2c} \quad \sqrt{T_1}\mathcal{L}_{1c} \oplus \sqrt{T_2}\mathcal{L}_{2c}.$

Thus automorphisms of type (i) generate real forms $\mathcal{L}_1 \oplus \mathcal{L}_2$ where $\mathcal{L}_i$ is $\sqrt{T_i}\,\mathcal{L}_{ic}$ $(i=1,2)$ which are semi-simple but not simple.

(b) Automorphisms of type (ii)

The involutive condition applied to $Z.\hat{T}_1.\hat{T}_2$ is satisfied if and only if $T_1 = T_2^{-1}$. Thus T_1 and T_2 are either both inner automorphisms or both outer automorphisms. For $T_2 = T_1^{-1}$, $Z.\hat{T}_1.\hat{T}_2\ (\underset{\sim}{a} \oplus \underset{\sim}{b})$ is $T_1^{-1}\underset{\sim}{b} \oplus T_1\underset{\sim}{a} = \hat{T}_1^{-1}.Z.\hat{T}_1\ (\underset{\sim}{a} \oplus \underset{\sim}{b})$, for all $\underset{\sim}{a} \oplus \underset{\sim}{b}$ in $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$.
So all involutive automorphisms of type (ii) are expressible in the form $W^{-1}.Z.W$ (where W is $\hat{T}_1$), and so condition N(3) of Cornwell [9] applies. Thus all automorphisms of type (ii) generate isomorphic real forms. It is therefore sufficient to take the simplest case in which T_1 is the identity, and examine $\sqrt{Z}\,(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$. Taking the generators of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ to be $\hat{\underset{\sim}{x}}^{(1)}_j$ and $\hat{\underset{\sim}{x}}^{(2)}_j$, as defined in section 2.2.1, Z has the form $\underset{\sim}{L}(2N)$, where N is the dimension of $\mathcal{L}_{1c}$ and $\mathcal{L}_{2c}$ and where $\underset{\sim}{L}(m)$ is defined for even m as

$$\underset{\sim}{L}(m) = \begin{pmatrix} \underset{\sim}{0}(\frac{1}{2}m) , & \underset{\sim}{I}(\frac{1}{2}m) \\ \underset{\sim}{I}(\frac{1}{2}m) , & \underset{\sim}{0}(\frac{1}{2}m) \end{pmatrix} \qquad (2.1)$$

Thus the generators of $\sqrt{Z}\,(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$ can be taken as $\underset{\sim}{X}^{(1)}_i = \hat{\underset{\sim}{x}}^{(1)}_i + \hat{\underset{\sim}{x}}^{(2)}_i$ and $\underset{\sim}{X}^{(2)}_i = i(\hat{\underset{\sim}{x}}^{(1)}_i - \hat{\underset{\sim}{x}}^{(2)}_i)$ for $i = 1,\ldots,N$, which obey the commutation relations

$$[\underset{\sim}{X}_i^{(1)}, \underset{\sim}{X}_j^{(1)}] = \sum_{k=1}^{M} c_{ij}^k \underset{\sim}{X}_k^{(1)};$$

$$[\underset{\sim}{X}_i^{(1)}, \underset{\sim}{X}_j^{(2)}] = \sum_{k=1}^{M} c_{ij}^k \underset{\sim}{X}_k^{(2)};$$

$$[\underset{\sim}{X}_i^{(2)}, \underset{\sim}{X}_j^{(2)}] = -\sum_{k=1}^{M} c_{ij}^k \underset{\sim}{X}_k^{(1)} \quad (i,j = 1,\ldots M)$$

where the c_{ij}^k are the structure constants of $\mathcal{L}_{1c}$. $\sqrt{Z}$ $(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$ is thus a simple real Lie algebra associated with a simple group G(C). Examples of Lie groups of the type G(C) are SL(ℓ+1, C) with $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$, SO(2$\ell$+1, C) with $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = B_\ell$, SO(2$\ell$, C) with $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = D_\ell$, and Sp(ℓ, C) with $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = C_\ell$.

2.2.4 The form of the automorphism Z

First taking the Lie algebra $\mathcal{L}_c$ with generators $\underset{\sim}{X}_j^{(1)}$ and $\underset{\sim}{X}_k^{(2)}$ (as defined in section 2.2.1), the definition of Z implies $Z(\underset{\sim}{a} \otimes \underset{\sim}{I}) = (\underset{\sim}{I} \otimes \underset{\sim}{a})$ and $Z(\underset{\sim}{I} \otimes \underset{\sim}{a}) = (\underset{\sim}{a} \otimes \underset{\sim}{I})$ for all $\underset{\sim}{a}$ in $\mathcal{L}_{1c}$. If Z is expressible as $T(\underset{\sim}{w})$ for some $\underset{\sim}{w}$, then $\underset{\sim}{w}$ can be shown to have elements of the form $w_{abcd} = k\delta_{ad}\,\delta_{bc}$ (a,b,c,d = 1,...M),

the index notation being as in a direct product, and k being a constant. $\underset{\sim}{w}$ is an N^2 dimensional matrix with N diagonal elements non-zero (the iiii th elements), and it may be completely diagonalized by the interchange of the ij th and ji th rows ($i \neq j$; i,j = 1,...N). After diagonalization the matrix obtained is $k\,\underset{\sim}{I}(N^2)$, which has determinant k^y, where $y = N^2$. Since there are $x = \frac{1}{2}(N^2 - N)$ interchanges in the diagonalization, the determinant of $\underset{\sim}{w}$ is given by det $\underset{\sim}{w}$ = $(-1)^x\, k^y$. For obvious reasons it may be convenient to choose k such that $\underset{\sim}{w}$ is special, (i.e. det $\underset{\sim}{w}$ = 1), orthogonal or unitary. For $\underset{\sim}{w}$ to be orthogonal, k must be ±1; for $\underset{\sim}{w}$ to be unitary, $k\,k^* = 1$, which implies $k = \exp(i\omega)$, where ω is real. k can always be chosen such that $\underset{\sim}{w}$ is unitary, orthogonal, and special, provided N mod 4 is not 2. In this case $\underset{\sim}{w}$ can not be both special and orthogonal.

Secondly, taking the generators of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$ to be $\hat{x}_j^{(1)}$ and $\hat{x}_k^{(2)}$ (as defined in section 2.2.1), the definition of Z implies

$$Z(\underset{\sim}{a} \oplus \underset{\sim}{0}) = (\underset{\sim}{0} \oplus \underset{\sim}{a}) \text{ and } Z(\underset{\sim}{0} \oplus \underset{\sim}{a}) = (\underset{\sim}{a} \oplus \underset{\sim}{0}) \quad (i = 1,\ldots N) \text{ where } \underset{\sim}{0} = \underset{\sim}{0}(N).$$

If Z can be expressed as $T(\underset{\sim}{w})$ for some $\underset{\sim}{w}$, then $\underset{\sim}{w}$ can be shown to be of the form

$$\begin{pmatrix} \underset{\sim}{0}(N) & , & c\underset{\sim}{I}(N) \\ c'\underset{\sim}{I}(N) & , & \underset{\sim}{0}(N) \end{pmatrix}$$

where c and c′ are arbitrary constants. If c and c′ are chosen to be -1 and 1 respectively,

$$\underset{\sim}{w} = \begin{pmatrix} \underset{\sim}{0}(N) & , & \underset{\sim}{I}(N) \\ -\underset{\sim}{I}(N) & , & \underset{\sim}{0}(N) \end{pmatrix}$$

and is special, orthogonal and unitary.

2.3. The embedding of real forms of $\tilde{\mathcal{L}}'$ in real forms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$, $\tilde{\mathcal{L}}'$, $\tilde{\mathcal{L}}_1$, and $\tilde{\mathcal{L}}_2$ all being simple

2.3.1 Basic theory

The embedding representation of $\mathcal{L}'_c$ will be denoted by

$$\underset{\sim}{\Gamma}(\mathcal{L}'_c) = \underset{\sim}{\rho}_1(\mathcal{L}'_c) \oplus \underset{\sim}{\rho}_2(\mathcal{L}'_c),$$

where $\underset{\sim}{\rho}_i(\underset{\sim}{a})$ belongs to $\mathcal{L}_{ic}$ for all $\underset{\sim}{a}$ in $\mathcal{L}'_c$ (i = 1,2). The generators $\underset{\sim}{a}'_i$ (i = 1,...m) must satisfy the condition

$$[\underset{\sim}{\Gamma}(\underset{\sim}{a}'_i), \underset{\sim}{\Gamma}(\underset{\sim}{a}'_j)] = \underset{\sim}{\Gamma}([\underset{\sim}{a}'_i, \underset{\sim}{a}'_j]) \qquad (j,i = 1,\ldots m),$$

which they do if and only if $[\underset{\sim}{\rho}_1(\underset{\sim}{a}'_i), \underset{\sim}{\rho}_1(\underset{\sim}{a}'_j)] = \underset{\sim}{\rho}_1([\underset{\sim}{a}'_i, \underset{\sim}{a}'_j])$ and $[\underset{\sim}{\rho}_2(\underset{\sim}{a}'_i), \underset{\sim}{\rho}_2(\underset{\sim}{a}'_j)] = \underset{\sim}{\rho}_2([\underset{\sim}{a}'_i, \underset{\sim}{a}'_j])$. Thus $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2$ are both representations of the simple Lie algebra $\mathcal{L}'_c$ and hence each is either a faithful representation or is a trivial representation by theorem 3 Appendix 3. For Γ to be faithful either ρ_1 or ρ_2 or both must be faithful. If either ρ_1 or ρ_2 is the trivial representation, then this is equivalent to embedding a simple Lie algebra in a simple Lie algebra which has been dealt with in [10, 11, 12] and in chapter 3. Both ρ_1 and ρ_2 will therefore be taken as faithful representaions in the following analysis.

An Automorphism S' of $\mathcal{L}'_c$ can be extended in the following way

$$S'_{ext} = \hat{S}'_{ext1} \cdot \hat{S}'_{ext2} \quad ,$$

where S'_{exti} $(i=1,2)$ is the extension of S' into $\mathcal{L}_{ic}$ with representation ρ_i. In the embedding condition (1.2), Y is such that $Y\,\Gamma(\mathcal{L}'_c) = \Gamma(\mathcal{L}'_c)$, which implies that

$$Y\,(\rho_1(\mathcal{L}'_c) \oplus \rho_2(\mathcal{L}'_c)) = \rho_1(\mathcal{L}'_c) \oplus \rho_2(\mathcal{L}'_c) .$$

Thus Y can be of either type (i) or type (ii). If of type (i), it must be of the form $Y = \hat{Y}_1 . \hat{Y}_2$, where $Y_i\,\rho_i(\mathcal{L}'_c) = \rho_i(\mathcal{L}'_c)$, $(i = 1,2)$. If Y is of type (ii) then it is of the form $Y = Z . \hat{Y}^{(1)} . \hat{Y}^{(2)}$ where $Y^{(1)}\rho_1(\underset{\sim}{a}) = \rho_2(\underset{\sim}{a})$ and $Y^{(2)}\rho_2(\underset{\sim}{a}) = \rho_1(\underset{\sim}{a})$, (for $\underset{\sim}{a}$ in $\mathcal{L}'_c$), which implies that ρ_1 and ρ_2 are equivalent, or in the case of $\mathcal{L}_i = A_\ell$ or E_6 $(i=1,2)$ ρ_1 and ρ_2 or ρ_1 and ρ^*_2 are equivalent. As S'_{ext} is of type (i), there are only two cases to be considered.

(a) <u>S of type (i)</u>

Y will also be of type (i) and so the embedding condition (1.2) becomes

$$\hat{S}_1.\hat{S}_2 = \hat{S}'_{ext1}.\hat{S}'_{ext2}.\hat{Y}_1.\hat{Y}_2$$

which is equivalent to $S_1 = S'_{ext1}.Y_1$ and $S_2 = S'_{ext2}.Y_2$. These are the embedding conditions(I.2) for embedding $\mathcal{L}'$ in $\mathcal{L}_1$ and $\mathcal{L}_2$ separately. Hence a necessary and sufficient condition for embedding $\mathcal{L}'$ in $\mathcal{L}_1 \oplus \mathcal{L}_2$ with embedding representation $\underset{\sim}{\rho}_1 \oplus \underset{\sim}{\rho}_2$ is that $\mathcal{L}'$ can be embedded in $\mathcal{L}_1$ and in $\mathcal{L}_2$ with representations $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2$ respectively.

(b) S of type (ii)

Y is of type (ii) and so the embedding condition(I.2)becomes

$$\hat{S}_1^{-1}.Z.\hat{S}_1 = \hat{S}'_{ext1}.\hat{S}'_{ext2}.Z.\hat{Y}^{(1)}.\hat{Y}^{(2)}.$$

The trace of both sides is zero, and since the left-hand side is involutive, the right-hand side must also be, which implies

$$\left.\begin{aligned} &S'_{ext1}.Y^{(2)}.S'_{ext2}.Y^{(1)} = I, \\ \text{and} \qquad &S'_{ext2}.Y^{(1)}.S'_{ext1}.Y^{(2)} = I. \end{aligned}\right\} \qquad (2.2)$$

It is convenient to consider the representation in the form

$$\underset{\sim}{\Gamma}(\mathcal{L}'_c) = (\underset{\sim}{B}_1 \oplus \underset{\sim}{B}_2)(\underset{\sim}{\rho}_1^c(\mathcal{L}'_c) \oplus \underset{\sim}{\rho}_2^c(\mathcal{L}'_c))(\underset{\sim}{B}_1 \oplus \underset{\sim}{B}_2)^{-1}$$

where $\underset{\sim}{\rho}_1^c$ and $\underset{\sim}{\rho}_2^c$ are in canonical form and completely reduced. The cases where $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2$ are equivalent and where $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2{}^*$ are equivalent will be considered separately.

(i) $\underset{\sim}{\rho}_1$ equivalent to $\underset{\sim}{\rho}_2$

In this case $\underset{\sim}{\rho}_1^c$ and $\underset{\sim}{\rho}_2^c$ are equal, and thus $Y^{(1)}$ and $Y^{(2)}$ must be of the form $Y^{(1)} = T(\underset{\sim}{B}_2)Y^c\,T(\underset{\sim}{B}_1^{-1})$, $Y^{(2)} = T(\underset{\sim}{B}_1)Y^c\,T(\underset{\sim}{B}_2^{-1})$, where $Y^c\underset{\sim}{\rho}_1^c = \underset{\sim}{\rho}_1^c$. Similarly S'_{exti} of the form $T(\underset{\sim}{B}_i)S^c\,T(\underset{\sim}{B}_i^{-1})$ $(i = 1,2)$, where S^c is the extension of S'

into $\mathcal{L}_{1c}$ using the representation $\underset{\sim}{\rho}_1^c$. Equation(2.2) then simplifies to

$$(S^c . Y^c)^2 = I. \tag{2.3}$$

(ii) $\underset{\sim}{\rho}_1$ equivalent to $\underset{\sim}{\rho}_2^*$

(This is applicable only if $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$ or E_6). In this case $\underset{\sim}{\rho}_1^c$ and $\underset{\sim}{\rho}_2^{c*}$ are equal. Thus $Y^{(1)}$ and S'_{exti} $(i = 1,2)$ must be of the form

$$Y^{(1)} = T(\underset{\sim}{B}_2)\ K.Y^c.T(\underset{\sim}{B}_1^{-1});\ Y^{(2)} = T(\underset{\sim}{B}_1).Y^c K.T(\underset{\sim}{B}_2^{-1});\ S'_{ext1} = T(\underset{\sim}{B}_1).S^c.T(\underset{\sim}{B}_1^{-1});$$
$$S'_{ext2} = T(\underset{\sim}{B}_2) K S^c K T(\underset{\sim}{B}_2^{-1}).$$

Equation (2.2) then again simplifies to (2.3), as in the case (i) where $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2$ are equivalent.

The condition (2.3) can always be satisfied by a suitable choice of Y^c, provided $\mathcal{L}'$ can be embedded in some real form of $\mathcal{L}_1$. Thus a necessary and sufficient condition for $\mathcal{L}'$ to be embeddable in $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$ is that $\mathcal{L}'$ can be embedded in some real form of $\tilde{\mathcal{L}}_1$ with representation $\underset{\sim}{\rho}_1$. The only possible forms of embedding representation are $\underset{\sim}{\rho}_1 \oplus \underset{\sim}{\rho}_1$ or $\underset{\sim}{\rho}_1 \oplus \underset{\sim}{\rho}_1^*$, the latter possibility only being permitted in the case $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$ or E_6. The generators of $\mathcal{L}' = \sqrt{S'}\mathcal{L}'_c$ are the same linear combination of $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}) = \sqrt{S'_{ext}Y}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$ as the generators of $\mathcal{L}'_c$ are of $(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$.

2.3.2 Example: The embedding of real forms of A_1 in real forms of $A_2 \oplus A_2$.

There are only two non-conjugate embeddings of SU(2) in SU(3), and thus there are three non-conjugate embeddings of SU(2) in SU(3) $\otimes$ SU(3) namely:

I) $\underset{\sim}{\Gamma}(\underset{\sim}{g}') \equiv (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}')) \otimes (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}'))$,

II) $\underset{\sim}{\Gamma}(\underset{\sim}{g}') \equiv \underset{\sim}{\Gamma}_3(\underset{\sim}{g}') \otimes \underset{\sim}{\Gamma}_3(\underset{\sim}{g}')$,

III) $\underset{\sim}{\Gamma}(\underset{\sim}{g}') \equiv (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}')) \otimes \underset{\sim}{\Gamma}_3(\underset{\sim}{g}')$,

where $\underset{\sim}{\Gamma}_n$ denotes the n-dimensional irreducible representation of SU(2).

Results for all possible embeddings of real forms of A_1 in real forms of A_2 are given in Cornwell [10] section 7.1 and in [11] section 6.1 : the representation $\underset{\sim}{\Gamma}_2(\underset{\sim}{g}') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}')$ provides embeddings of SU(2) in SU(3) and SU(2,1) and of SU(1,1) in SU(2,1) and SL(3,R) ; the representation $\underset{\sim}{\Gamma}_3(\underset{\sim}{g}')$ provides for embeddings of SU(2) in SU(3) and SL(3,R) and of SU(1,1) in SU(2,1) and SL(3,R) . Using this information all embeddings of real forms of A_1 in real forms of $A_2 \oplus A_2$ except SL(3,C) can be calculated. Since representations I and II satisfy $\rho_1^c = \rho_2^c$, and representation III satisfies neither $\rho_1^c = \rho_2^c$ nor $\rho_1^c = \rho_2^{c*}$, SU(2) and SU(1,1) can be embedded in SL(3,C) with representations I and II , but not with representation III. These results are summarised in table 2.1 .

Table 2.1 : Embeddings of real forms of A_1 in those of $A_2 \oplus A_2$

Embeddings exist with the representations whose numbers are indicated.

Real forms of $A_2 \oplus A_2$	Real forms of A_1	
	$SU(2) = Q_1$	$SU(1,1) = SL(2,R)$
$SU(3) \otimes SU(3)$	I,II,III	none
$SU(3) \otimes SU(2,1)$	I	none
$SU(3) \otimes SL(3,R)$	II,III	none
$SU(2,1) \otimes SU(3)$	I,III	none
$SU(2,1) \otimes SU(2,1)$	I	I,II,III
$SU(2,1) \otimes SL(3,R)$	III	I,II,III
$SL(3,R) \otimes SU(3)$	II	none
$SL(3,R) \otimes SU(2,1)$	none	I,II,III
$SL(3,R) \otimes SL(3,R)$	II	I,II,III
$SL(3,C)$	I,II	I,II

NB. As discussed in the text, the relatively simple case corresponding to embeddings of real forms of A_1 in real forms of A_2 alone is not included.

2.4 Embeddings of $\mathcal{L}_1' \oplus \mathcal{L}_2'$ in $\mathcal{L}_1 \oplus \mathcal{L}_2$, the extension of Z and the centralizer.

2.4.1 Possible types of embedding

The general form of the embedding representation can be expressed as :

$\Gamma(\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}') = (\rho_{11}(\mathcal{L}_{1c}') + \rho_{21}(\mathcal{L}_{2c}')) \oplus (\rho_{12}(\mathcal{L}_{1c}') + \rho_{22}(\mathcal{L}_{2c}'))$,

where $\rho_{ij}(\mathcal{L}_{ic}')$ is contained in $\mathcal{L}_{jc}$ ($i,j = 1,2$) . Since $\mathcal{L}_{1c}' \oplus 0$ and $0 \oplus \mathcal{L}_{2c}'$ commute , it follows that ρ_{1j} and ρ_{2j} $(j=1,2)$ commute . The representation Γ must satisfy

$\underset{\sim}{\Gamma}([\underset{\sim}{a}_1' \oplus \underset{\sim}{b}_1' , \underset{\sim}{a}_2' \oplus \underset{\sim}{b}_2']) = [\underset{\sim}{\Gamma}(\underset{\sim}{a}_1' \oplus \underset{\sim}{b}_1'), \underset{\sim}{\Gamma}(\underset{\sim}{a}_2' \oplus \underset{\sim}{b}_2')]$ for all $\underset{\sim}{a}_1' \oplus \underset{\sim}{b}_1'$ and $\underset{\sim}{a}_2' \oplus \underset{\sim}{b}_2'$ in $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$, and hence the ρ_{ij} satisfy

$\rho_{ij}([\underset{\sim}{a}, \underset{\sim}{b}]) = [\rho_{ij}(\underset{\sim}{a}), \rho_{ij}(\underset{\sim}{b})]$ $(i,j=1,2)$,which implies that ρ_{ij} is a representation of $\mathcal{L}_{ic}'$ in $\mathcal{L}_{jc}$. Some of the ρ_{ij} may be trivial representations and hence there are four distinct types of embedding .

Embedding 1.

In this embedding only ρ_{11} and ρ_{22} are non-trivial and thus $\Gamma(\mathcal{L}_{1c}' \oplus 0)$ is a subalgebra of $\mathcal{L}_{1c} \oplus 0$ and $\Gamma(0 \oplus \mathcal{L}_{2c}')$ a subalgebra of $0 \oplus \mathcal{L}_{2c}$. ρ_{12} and ρ_{21} being the only non-trivial representations gives an embedding of the same type .

Embedding 2.

In this case ρ_{11} and ρ_{21} are the only non-trivial representations , and thus $\Gamma(\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}')$ is a subalgebra of $\mathcal{L}_{1c} \oplus 0$. This case is equivalent to the embedding of $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$ in $\mathcal{L}_{1c}$. ρ_{12} and ρ_{22} being the only non-trivial representations gives a similar embedding .

Embedding 3.

All the ρ_{ij} are faithful in this case .

Embedding 4.

In this case ρ_{21} is the only trivial representation . Any of the other ρ_{ij} being the only trivial representation give embeddings of the same form .

2.4.2 The extension of Z' for the above embeddings.

It is implicit in this section that $\mathcal{L}'_1$ is isomorphic to $\mathcal{L}'_2$.

Embedding 1

From its definition Z'_{ext} must satisfy

$$Z'_{ext}(\rho_{11}(\mathcal{L}'_{1c}) \oplus \rho_{22}(\mathcal{L}'_{2c})) = \rho_{11}(\mathcal{L}'_{2c}) \oplus \rho_{22}(\mathcal{L}'_{2c}) \text{ ,}$$

and hence Z'_{ext} is obviously of type (ii) and can be written $Z'_{ext} = Z.\hat{Z}'_1.\hat{Z}'_2$ where $Z'_1\,\rho_{11} = \rho_{22}$ and $Z'_2\,\rho_{22} = \rho_{11}$. Clearly this is only possible if $\tilde{\mathcal{L}}_1$ is isomorphic to $\tilde{\mathcal{L}}_2$. Thus $Z'_1.Z'_2\,\rho_{22} = \rho_{22}$ and $Z'_2.Z'_1\,\rho_{11} = \rho_{11}$, so Z'_2 can be taken to be Z'^{-1}_1 , any additional factor being taken up in the centralizer. Thus for the existance of Z'_{ext} , ρ_{11} must be equivalent to ρ_{22} or ρ^*_{22} , the latter possibility being permitted only if $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$ or E_6 .When expressed in canonical form $\underset{\sim}{T}(\underset{\sim}{a}' \oplus \underset{\sim}{b}') = (\underset{\sim}{B}_1 \oplus \underset{\sim}{B}_2)\ (\underset{\sim}{\rho}^c_{11}(\underset{\sim}{a}') \oplus \underset{\sim}{\rho}^c_{22}(\underset{\sim}{b}'))(\underset{\sim}{B}_1 \oplus \underset{\sim}{B}_2)^{-1}$, (the $\underset{\sim}{\rho}^c_{ii}$ being completely reduced (i=1,2) . Then ρ^c_{11} must be equal to ρ^c_{22} or ρ^{c*}_{22}, the latter possibility only being allowed for $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$ or E_6. If ρ^c_{11} and ρ^c_{22} are equal then $Z'_1 = T(\underset{\sim}{B}_2.\underset{\sim}{B}^{-1}_1)$ and $Z'_2 = T(\underset{\sim}{B}_1.\underset{\sim}{B}^{-1}_2)$. If ρ^c_{11} and ρ^{c*}_{22} are equal then $Z'_1 = T(\underset{\sim}{B}_2).K.T(\underset{\sim}{B}^{-1}_1)$ and $Z'_2 = T(\underset{\sim}{B}_1).K.T(\underset{\sim}{B}^{-1}_2)$.

Embedding 2

Z'_{ext} must satisfy

$$Z'_{ext}(\rho_{11}(\mathcal{L}'_{1c}) + \rho_{21}(\mathcal{L}'_{2c})) = \rho_{11}(\mathcal{L}'_{2c}) + \rho_{21}(\mathcal{L}'_{1c}) \text{ ,}$$

and hence ρ_{11} is equivalent to ρ_{21} or to ρ^*_{21} (for $\tilde{\mathcal{L}}_1 = A_\ell$

or E_6 only) . These cases are now examined separately.

a) ρ_{11} equivalent to ρ_{21}.

The representation may be written as

$$\underset{\sim}{\Gamma}(\underset{\sim}{g}_1 \otimes \underset{\sim}{g}_2) = \underset{\sim}{B}(\oplus\sum_j \Gamma^a_j(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) \quad \oplus\sum_{kk'} \underset{\sim}{\Gamma}^b_{kk'}(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2))\underset{\sim}{B}^{-1} , \qquad (2.4)$$

where $\underset{\sim}{\Gamma}^a_j(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) = \underset{\sim}{\Gamma}^j(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\Gamma}^j(\underset{\sim}{g}'_2)$, and

$$\underset{\sim}{\Gamma}^b_{kk'}(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) = (\underset{\sim}{\Gamma}^k(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\Gamma}^{k'}(\underset{\sim}{g}'_2)) \oplus (\underset{\sim}{\Gamma}^{k'}(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\Gamma}^k(\underset{\sim}{g}'_2)) , \quad (k \neq k')$$

$\underset{\sim}{\Gamma}^j$ being the jth irreducible representation occurring in the reduction of $\underset{\sim}{\Gamma}$, It is possible for some of the $\underset{\sim}{\Gamma}^j$ to be the trivial representation. It is convenient to introduce the matrix $\underset{\sim}{w}(n_p, n_q)$, whose elements are $w_{ijkl} = \delta_{il}\,\delta_{jk}$ where $i,l=1,..n_p$ and $j,k=1,...,n_q$, the index notation being as for a direct product .(This is a generalization of the first $\underset{\sim}{w}$ of section 2.2.4) . Then with $\underset{\sim}{W}(n_p, n_q)$ defined as

$$\underset{\sim}{W}(n_p, n_q) = \begin{pmatrix} \underset{\sim}{O}(n_p.n_q) & , \underset{\sim}{w}(n_q, n_p) \\ \underset{\sim}{w}(n_p, n_q) & , \underset{\sim}{O}(n_p.n_q) \end{pmatrix}$$

Z' can be extended to give

$$Z'_{ext} \equiv T(\underset{\sim}{B}).T(\oplus \sum_j \underset{\sim}{W}(n_j, n_j) \quad \oplus \sum_{kk'} \underset{\sim}{W}(n_k, n_{k'})).T(\underset{\sim}{B})^{-1} \qquad (2.5)$$

$$\equiv T(\underset{\sim}{w}'_{ext}) ,$$

where n_j denotes the dimension of Γ^j.

b) ρ_{11} equivalent to ρ^*_{21}.

This case only applies for $\tilde{\mathcal{L}}_1 = A_\ell$ or E_6 ,embeddings being of the form .

$$\underset{\sim}{\Gamma}(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) = \underset{\sim}{B}(\oplus \sum_j \underset{\sim}{\Gamma}^c_j(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) \quad \oplus \sum_{kk'} \underset{\sim}{\Gamma}^d_{kk'}(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2))\underset{\sim}{B}^{-1} , \qquad (2.6)$$

where $\underset{\sim}{\Gamma}^c_j(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) = \underset{\sim}{\Gamma}^j(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\Gamma}^j(\underset{\sim}{g}_2{}')$,

$\underset{\sim}{\Gamma}^d_{kk'}(\underset{\sim}{g}'_1 \otimes \underset{\sim}{g}'_2) = (\underset{\sim}{\Gamma}^k(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\Gamma}^{k'}(\underset{\sim}{g}'_2) \quad \oplus \quad (\underset{\sim}{\Gamma}^{k'}(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\Gamma}^k(\underset{\sim}{g}'_2))$.

Z' can then be extended to give

$$Z'_{ext} \equiv T(\underset{\sim}{w}'_{ext}).K_{ext} \tag{2.7}$$

where $\underset{\sim}{w}'_{ext}$ is defined for the case of ρ_{11} equivalent to ρ_{21} and

$K_{ext} = T(\underset{\sim}{B}).K.T(\underset{\sim}{B})$.

Embedding 3

Z'_{ext} must satisfy

$Z'_{ext}((\rho_{11}(\mathcal{L}'_{1c}) + \rho_{21}(\mathcal{L}'_{2c})) \oplus (\rho_{12}(\mathcal{L}'_{1c}) + \rho_{22}(\mathcal{L}'_{2c})))$
$= (\rho_{11}(\mathcal{L}'_{2c}) + \rho_{21}(\mathcal{L}'_{1c})) \oplus (\rho_{12}(\mathcal{L}'_{2c}) + \rho_{22}(\mathcal{L}'_{1c}))$.

and can thus be of either type (i) or type (ii) .

If of type (i) , ρ_{1i} and ρ_{2i} must be equivalent $(i=1,2)$ or in the case of $\tilde{\mathcal{L}}_i = A_\ell$ or E_6 , ρ_{1i} and ρ^*_{2i} may be equivalent . Z'_{ext} can then be taken as $\hat{Z}'_{ext1}.\hat{Z}'_{ext2}$, where $Z'_{exti}(i=1,2)$ is the extension of Z' into $\mathcal{L}_i$ for an embedding of type 2 . Thus $Z'_{exti} = T(\underset{\sim}{w}'_{exti})$ or $T(\underset{\sim}{w}'_{exti}).K_{exti}$.

A Z'_{ext} of type (ii) exists if and only if ρ_{11} is equivalent to ρ_{22} and ρ_{12} is equivalent to ρ_{21} , or in the case where $\tilde{\mathcal{L}}_i$ is A_ℓ or E_6 , ρ_{22} and ρ_{21} may both be replaced by their complex conjugates. Γ can thus be expressed in the form

$\underset{\sim}{\Gamma}(\underset{\sim}{a}'\oplus\underset{\sim}{b}') = (\underset{\sim}{B}_1\oplus\underset{\sim}{B}_2)((\underset{\sim}{\rho}^c_{11}(\underset{\sim}{a}') + \underset{\sim}{\rho}^c_{21}(\underset{\sim}{b}')) \oplus (\underset{\sim}{\rho}^c_{12}(\underset{\sim}{a}') + \underset{\sim}{\rho}^c_{22}(\underset{\sim}{b}')))(\underset{\sim}{B}_1\oplus\underset{\sim}{B}_2)^{-1}$

where $\rho^c_{11} = \rho^c_{22}$ or ρ^{c*}_{22} and $\rho^c_{12} = \rho^c_{21}$ or ρ^{c*}_{21} , and the ρ^c_{ij} are in canonical form and completely reduced . Z'_{ext} is then given by $Z'_{ext} = Z.\hat{Z}'_1.\hat{Z}'_2$, where $Z'_1 = T(\underset{\sim}{B}_2\underset{\sim}{B}_1^{-1})$ and $Z'_2 = T(\underset{\sim}{B}_1\underset{\sim}{B}_2^{-1})$ if $\rho^c_{11} = \rho^c_{22}$ and $\rho^c_{12} = \rho^c_{21}$ or $Z'_1 = T(\underset{\sim}{B}_2).K.T(\underset{\sim}{B}_1^{-1})$ and $Z'_2 = T(\underset{\sim}{B}_1).K.T(\underset{\sim}{B}_2)$ if $\rho^c_{11} = \rho^{c*}_{22}$ and $\rho^c_{12} = \rho^{c*}_{21}$.

Embedding 4

Z'_{ext} must satisfy

$$Z'_{ext}(\rho_{11}(\mathcal{L}'_{1c}) \oplus (\rho_{12}(\mathcal{L}'_{1c}) + \rho_{22}(\mathcal{L}'_{2c}))) = \rho_{11}(\mathcal{L}'_{2c}) \oplus (\rho_{12}(\mathcal{L}'_{2c}) + \rho_{22}(\mathcal{L}'_{1c})).$$

Since there is no automorphism of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ which satisfies this condition , there is no extension for Z', and hence no embeddings of type 4 for $\sqrt{Z'}(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c})$.

2.4.3 The form of the centralizer for embeddings 1,2,3 and 4.

The centralizer $C(G'_c , G_c)$ for the embedding of G'_c in G_c (G'_c and G_c both being simple) consists of elements $\underset{\sim}{c}$ which can be written in the form $\underset{\sim}{c} = \underset{\sim}{r}.\exp \underset{\sim}{g} .\underset{\sim}{r}^{-1}$, where $\underset{\sim}{g}$ is a chief centralizer generator, (Cornwell [8]). When G'_c and G_c are semi-simple , being the direct product of at most two simple compact Lie groups , the centralizer elements may also be written in a canonical form. This form will now be investigated .

Embedding 1

The embedding representation is of the form $\underset{\sim}{\Gamma}(g'_1 \otimes g'_2) = \underset{\sim}{\rho}_{11}(\underset{\sim}{g}'_1) \otimes \underset{\sim}{\rho}_{22}(\underset{\sim}{g}'_2)$ where $\underset{\sim}{\rho}_{ii}(\underset{\sim}{g}'_i) \in G_{ic}$ $(i=1,2)$. A member of the centralizer must be of the form $\underset{\sim}{c}_1 \otimes \underset{\sim}{c}_2$, since it is a member of $G_{1c} \otimes G_{2c}$, and must satisfy $T(\underset{\sim}{c}_i)\underset{\sim}{\rho}_{ii}(\underset{\sim}{g}'_i) = \underset{\sim}{\rho}_{ii}(\underset{\sim}{g}'_i)$, $i=1,2$. Hence $\underset{\sim}{c}_i \in C(G'_{ic} , G_{ic})$. All members of the centralizer $C(G'_{1c} \otimes G'_{2c} , G_{1c} \otimes G_{2c})$ can thus be expressed as

$$\underset{\sim}{c}_1 \otimes \underset{\sim}{c}_2 = \underset{\sim}{r}_1.\exp \underset{\sim}{g}_1.\underset{\sim}{r}_1^{-1} \otimes \underset{\sim}{r}_2.\exp \underset{\sim}{g}_2.\underset{\sim}{r}_2^{-1} = \underset{\sim}{r}.\exp \underset{\sim}{g}.\underset{\sim}{r}^{-1} \quad \text{for}$$

$\underset{\sim}{r} = \underset{\sim}{r}_1 \otimes \underset{\sim}{r}_2$ and $\underset{\sim}{g} = \underset{\sim}{g}_1 \otimes \underset{\sim}{I} + \underset{\sim}{I} \otimes \underset{\sim}{g}_2$.

Embedding 2

The argument used by Cornwell [8] applies for G'_c being simple or semi-simple , and for G_c being simple , and so holds for embeddings of this type .

Embedding 3

The embedding representation is of the form

$$\underset{\sim}{\Gamma}(\underset{\sim}{g}_1'\otimes\underset{\sim}{g}_2') = (\underset{\sim}{\rho}_{11}(\underset{\sim}{g}_1') \times \underset{\sim}{\rho}_{21}(\underset{\sim}{g}_2'))\otimes(\underset{\sim}{\rho}_{12}(\underset{\sim}{g}_1') \times \underset{\sim}{\rho}_{22}(\underset{\sim}{g}_2')) \in G_{1c}\otimes G_{2c} ,$$

and hence the centralizer elements are of the form $\underset{\sim}{c}_1\otimes\underset{\sim}{c}_2$, where $T(\underset{\sim}{c}_i)(\underset{\sim}{\rho}_{1i}(\underset{\sim}{g}_1') \times \underset{\sim}{\rho}_{2i}(\underset{\sim}{g}_2')) = \underset{\sim}{\rho}_{1i}(\underset{\sim}{g}_1') \times \underset{\sim}{\rho}_{2i}(\underset{\sim}{g}_2')$ for $i=1,2$. Thus $\underset{\sim}{c}_i \in C(G_{1c}'\otimes G_{2c}' , G_{ic})$ for embedding representation $\rho_{1i}\times\rho_{2i}$ and so the centralizer elements can be expressed in the form

$$\underset{\sim}{c}_1\otimes\underset{\sim}{c}_2 = \underset{\sim}{r}_1\circ\exp \underset{\sim}{g}_1\circ\underset{\sim}{r}_1^{-1} \otimes \underset{\sim}{r}_2\circ\exp \underset{\sim}{g}_2\circ\underset{\sim}{r}_2^{-1} = \underset{\sim}{r}\circ\exp \underset{\sim}{g}\circ\underset{\sim}{r}^{-1} .$$

Embedding 4

This case follows through as for embedding 3 with $\rho_{21}=0$, and so the centralizer is of the same form .

Thus the centralizer elements for embeddings 1,2,3 and 4 can all be expressed in the canonical form $\underset{\sim}{c} = \underset{\sim}{r}\circ\exp \underset{\sim}{g}\circ\underset{\sim}{r}^{-1}$.

2.5 Embedding 1.

2.5.1 Basic Theory.

The general form of the embedding representation is

$\Gamma(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c}) = \rho_{11}(\mathcal{L}'_{1c}) \oplus \rho_{22}(\mathcal{L}'_{2c})$.

S and S' appearing in condition (1.2) , may either be of type (i) or type (ii) , and hence there is the possibility of Y being of type(i) or type (ii) also . (It may be recalled that from the definition of Y $\Upsilon.\underset{\sim}{\Gamma}(\underset{\sim}{a}' \oplus \underset{\sim}{b}') = \underset{\sim}{\Gamma}(\underset{\sim}{a}' \oplus \underset{\sim}{b}')$, for all $\underset{\sim}{a}' \oplus \underset{\sim}{b}' \in \mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c}$). A Y of type (ii) would be of the form $Y = Z.\hat{Y}^{(1)}.\hat{Y}^{(2)}$, where $Y^{(1)} \rho_{11}(\mathcal{L}'_{1c})$ is $\rho_{22}(\mathcal{L}'_{2c})$ and $Y^{(2)} \rho_{22}(\mathcal{L}'_{2c})$ is $\rho_{11}(\mathcal{L}'_{1c})$. Such $Y^{(i)}$ obviously do not exist , and so Y must be of type (i) . S and S'_{ext} must therefore both be of the same type , so there aretwo cases to consider:-

a) S'and S of type(i)

If S' is given by $\hat{S}'_1.\hat{S}'_2$, then its extension is given by $\hat{S}'_{1ext1}.\hat{S}'_{2ext2}$ where S'_{iexti} is the extension of S'_i into $\mathcal{L}_{ic}$ with the embedding representation ρ_{ii} . Taking Y as $\hat{Y}_1.\hat{Y}_2$, condition (1.2) becomes $\hat{S}_1.\hat{S}_2 = \hat{S}'_{1ext1}.\hat{S}'_{2ext2}.\hat{Y}_1.\hat{Y}_2$, which is equivalent to $S_1 = S'_{1ext1}.Y_1$ and $S_2 = S'_{2ext2}.Y_2$. These are the conditions for embedding $\mathcal{L}'_i$ in $\mathcal{L}_i$ (i=1,2) with representation ρ_{ii}.
Hence a necessary and sufficient condition for embedding $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ in $\mathcal{L}_1 \oplus \mathcal{L}_2$ with representation $\rho_{11} \oplus \rho_{22}$ is that $\mathcal{L}'_i$ can be embedded in $\mathcal{L}_i$ with representation ρ_{ii} (i=1,2).

b) S' and S of type (ii)

If S' is given by $Z'.\hat{S}'_1.\hat{S}'_2$ then its extension may be taken as $Z.\hat{Z}'_1.\hat{Z}'_2.\hat{S}'_{1ext1}.\hat{S}'_{2ext2}$ (see section 2.4.2). Assuming S is of the form $Z.\hat{S}_1.\hat{S}_2$ and Y is of the form $\hat{Y}_1.\hat{Y}_2$, the condition (1.2) will be examined for the two types of permitted embedding , namely ρ_{11} and ρ_{22} equivalent and ρ_{11} and ρ_{22}^* equivalent .

ρ_{11} and ρ_{22} equivalent

This implies that ρ^c_{11} and ρ^c_{22} are equal . In this case $Z_1'=T(\underset{\sim}{B}_2.\underset{\sim}{B}_1^{-1})$, $Z_2'=T(\underset{\sim}{B}_1.\underset{\sim}{B}_2^{-1})$, $S'_{iexti} = T(\underset{\sim}{B}_i).S^c_i.T(\underset{\sim}{B}_i^{-1})$ and $Y_i= T(\underset{\sim}{B}_i).Y^c_i.T(\underset{\sim}{B}_i^{-1})$, where $Y^c_i\ \rho_{ii}(\mathcal{L}'_i) = \rho_{ii}(\mathcal{L}'_i)$ and S^c_i is the extension of S'_i into $\mathcal{L}_1$ with embedding representation ρ^c_{i1} $(i=1,2)$. Thus embedding condition (1.2) becomes :

$Z.\widehat{S}_1.\widehat{S}_2 = Z.\widehat{Z}'_1.\widehat{Z}'_2.\widehat{S}'_{1ext1}.\widehat{S}'_{2ext2}.\widehat{Y}_1.\widehat{Y}_2$. This implies that $S_1=(Z'_1.S'_{1ext1}.Y_1)= S_2^{-1} = (Z'_2.S'_{2ext2}.Y_2)^{-1}$. Since $S_1 = S_2^{-1}$ is arbitrary this can be satisfied for

$$S^c_1.Y^c_1 = (S^c_2.Y^c_2)^{-1} \tag{2.8}$$

Since the choice of $S'_1 = S_2'^{-1}$ is arbitrary , this condition can always be satisfied for a suitable choice of S'_1 and Y_1 (for example $S'_i=Y_i=I$). Different choices of S'_i and Y_i provide conjugate embeddings .

ρ_{11} and ρ^*_{22} equivalent

This implies that ρ^c_{11} and ρ^{c*}_{22} are equal . In this case $Z_1'=T(\underset{\sim}{B}_2).K.T(\underset{\sim}{B}_1^{-1})$, $Z_2'=T(\underset{\sim}{B}_1).K.T(\underset{\sim}{B}_2^{-1})$, $S'_{1ext1}= T(\underset{\sim}{B}_1).S^c_1.T(\underset{\sim}{B}_1^{-1})$, $S'_{2ext2}= T(\underset{\sim}{B}_2).K.S^c_2.K.T(\underset{\sim}{B}_2^{-1})$, $Y_1 = T(\underset{\sim}{B}_1).Y^c_1.T(\underset{\sim}{B}_1^{-1})$ and $Y_2 = T(\underset{\sim}{B}_2).K.Y^c_2.K.T(\underset{\sim}{B}_2^{-1})$, where Y^c_i and S^c_i are as defined for ρ_{11} and ρ_{22} equivalent . Condition (1.2) again simplifies to give (2.8),and the argument follows as for ρ_{11} and ρ_{22} equivalent .

Thus a necessary and sufficient condition for an embedding of $\sqrt{Z}(\mathcal{L}'_{1c}\oplus\mathcal{L}'_{2c})$ in $\sqrt{Z}(\mathcal{L}_{1c}\oplus\mathcal{L}_{2c})$ with representation $\rho_{11}\oplus\rho_{22}$ is that ρ_{11} and ρ_{22} be equivalent or in the case of $\mathcal{L}_i= A_\ell$ or E_6 $(i=1,2)$ ρ_{11} and ρ^*_{22} be equivalent representations of $\mathcal{L}'_{1c}$ given that $\mathcal{L}'_{1c}\cong\mathcal{L}'_{2c}$ can be embedded in $\mathcal{L}_{1c}\cong\mathcal{L}_{2c}$ with representation ρ_{11} . The generators of the embedded subalgebra are the same linear combination of the generators of $\sqrt{Z}(\mathcal{L}_{1c}\oplus\mathcal{L}_{2c})$ as the generators of $\mathcal{L}'_{1c}\oplus\mathcal{L}'_{2c}$ are of $\mathcal{L}_{1c}\oplus\mathcal{L}_{2c}$.

2.5.2 <u>Example</u>: The embedding of real forms of $A_1 \oplus A_1$ in real forms of $A_2 \oplus A_2$.
There are three non-conjugate embeddings of $SU(2) \otimes SU(2)$ in $SU(3) \otimes SU(3)$, of type 1 , namely those with embedding representations :

I) $\underset{\sim}{\Gamma}(\underset{\sim}{g}_1' \otimes \underset{\sim}{g}_2') = (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}_1') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}_1')) \otimes (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}_2') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}_2'))$,

II) $\underset{\sim}{\Gamma}(\underset{\sim}{g}_1' \otimes \underset{\sim}{g}_2') = \underset{\sim}{\Gamma}_3(\underset{\sim}{g}_1') \otimes \underset{\sim}{\Gamma}_3(\underset{\sim}{g}_2')$

III) $\underset{\sim}{\Gamma}(\underset{\sim}{g}_1' \otimes \underset{\sim}{g}_2') = (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}_1') \oplus \underset{\sim}{\Gamma}_1(\underset{\sim}{g}_1')) \otimes \underset{\sim}{\Gamma}_3(\underset{\sim}{g}_2')$

From Cornwell [10] section 7.1 and [11] section 6.1 the following results are used : $SU(2)$ may be embedded in $SU(3)$ with representations $\Gamma_2 \oplus \Gamma_1$ and Γ_3 and in $SU(2,1)$ with representation $\Gamma_2 \oplus \Gamma_1$ and in $SL(3,R)$ with representation Γ_3 : $SU(1,1)$ can be embedded in $SU(2,1)$ and $SL(3,R)$ with both these representations. Thus embeddings of real forms of $A_1 \oplus A_1$,generated by automorphisms of type (i) can easily be found .

In representations I and II ρ_{11} and ρ_{22} are equal. Thus $SL(2,C)$ can be embedded in $SL(3,C)$ with both these representations . However in representation III ρ_{11} is equivalent to neither ρ_{22} nor ρ_{22}^*, so there are no embeddings of $SL(2,C)$ with this representation. All embeddings of real forms of $A_1 \oplus A_1$ in real forms of $A_2 \oplus A_2$ of type 1 are summarised in table 2.2 .

Table 2.2: Possible embeddings of real forms of $A_1 \oplus A_1$ in those of $A_2 \oplus A_2$ (with embedding 1)

Embeddings exist with the representations, whose numbers are indicated.

Real forms of $A_2 \oplus A_2$	Real forms of $A_1 \oplus A_1$				
	$SU(2) \otimes SU(2)$	$SU(1,1) \otimes SU(2)$	$SU(2) \otimes SU(1,1)$	$SU(1,1) \otimes SU(1,1)$	$SL(2,C)$
$SU(3) \otimes SU(3)$	I,II,III	none	none	none	none
$SU(3) \otimes SU(2,1)$	I	none	I,II,III	none	none
$SU(3) \otimes SL(3,R)$	II,III	none	I,II,III	none	none
$SU(2,1) \otimes SU(3)$	I,III	I,II,III	none	none	none
$SU(2,1) \otimes SU(2,1)$	I	I	I,III	I,II,III	none
$SU(2,1) \otimes SL(3,R)$	III	II,III	I,III	I,II,III	none
$SL(3,R) \otimes SU(3)$	II	I,II,III	none	none	none
$SL(3,R) \otimes SU(1,1)$	none	I	II	I,II,III	none
$SL(3,R) \otimes SL(3,R)$	II	II,III	II	I,II,III	none
$SL(3,C)$	none	none	none	none	I,II

2.6. Embedding 2.

In this case it may be assumed that $\mathcal{L}_1' \oplus \mathcal{L}_2'$ is a subalgebra of a simple Lie algebra $\mathcal{L}$.

2.6.1 S' inner

(a) S inner

Since the centralizer can be expressed in the form $\underset{\sim}{c}=\underset{\sim}{r}\circ\exp\underset{\sim}{g}\circ\underset{\sim}{r}^{-1}$ (section 2.4.3) , this case goes through in exactly the same way as the simple case of embedding a simple Lie algebra in a simple Lie algebra that was given in [10] section 6.

(b) S outer

There are two cases to consider , namely $\tilde{\mathcal{L}}=A_\ell$ (or E_6) and $\tilde{\mathcal{L}}=D_\ell$. As in [11] and [12] Y will be of the form $T(\underset{\sim}{c}\circ\underset{\sim}{y})\circ K$ for $\tilde{\mathcal{L}}=A_\ell$ (or E_6) and $T(\underset{\sim}{p}\circ\underset{\sim}{c})$ for $\tilde{\mathcal{L}}=D_\ell$, the definitions for $\underset{\sim}{y}$ and $\underset{\sim}{p}$ being given in [11] section 3(c) and [12] section 3(b) respectively (and in section 1.3).

For $\tilde{\mathcal{L}}=A_\ell$ (or E_6) the embedding condition becomes $S=T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\circ\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}\circ\underset{\sim}{c}\circ\underset{\sim}{y})\circ K_{ext}$, which is the same form as in the simple case , and hence the arguments and conditions of [11] section 3(c) apply .

For $\mathcal{L}=D_\ell$ the embedding condition becomes $S=T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\circ\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}\circ\underset{\sim}{p}\circ\underset{\sim}{c})$, and this too is of the same form as the simple case , the arguments and conditions being as given in [12] section 3(b) .

2.6.2 S' outer of type (i)

S' is of the form $S'=\hat{S}_1'\circ\hat{S}_2'$, where at least one of the S_i' (i=1,2) is outer . Thus $\tilde{\mathcal{L}}_1'$ and/or $\tilde{\mathcal{L}}_2'$ must be A_ℓ , E_6 or D_ℓ . The different possibilities for $\hat{S}_1'$, $\hat{S}_2'$, $\tilde{\mathcal{L}}_1'$, $\tilde{\mathcal{L}}_2'$ will be examined separately:

(a) S_1' and S_2' both outer , $\tilde{\mathcal{L}}_1'=A_\ell$(or E_6) and $\tilde{\mathcal{L}}_2'=A_\ell$ (or E_6)

S' is of the form $T(\underset{\sim}{u}')\circ T(\underset{\sim}{d}_1'\otimes\underset{\sim}{d}_2')\circ K_{ext}\circ\exp(ad(\underset{\sim}{h}'))\circ T(\underset{\sim}{u}')^{-1}$, which is of the same form as the simple case of embedding real forms of A_ℓ generated by outer automorphisms . The argument thus follows that given for the simple cases in section 3(b) and 3(d) in [11] (for embeddings in A_ℓ and E_6) and section 5(b) in [11](for embeddings in B_ℓ and C_ℓ) and section 6(c) and (d) (for embeddings in D_ℓ), $SL(\ell'+1,R) \otimes SL(\ell''+1,R)$ and $Q_{\frac{1}{2}(\ell'+1)} \otimes Q_{\frac{1}{2}(\ell''+1)}$ follow as for $SL(\ell'+1,R)$: and $SL(\ell'+1,R) \otimes Q_{\frac{1}{2}(\ell''+1)}$ and $Q_{\frac{1}{2}(\ell'+1)}\, SL(\ell''+1,R)$ follow as for $Q_{\frac{1}{2}(\ell'+1)}$.

b) <u>S_1' outer, S_2' inner</u> , <u>$\tilde{\mathcal{L}}_1' = A_\ell$ or E_6</u> .

S' is of the form $T(\underset{\sim}{u}')T(\underset{\sim}{d}_1' \otimes \underset{\sim}{I}).K_1.T(\exp \underset{\sim}{h}')T(\underset{\sim}{u}')^{-1}$, the extension of which exists if and only if either ρ_{11} or ρ_{21} is equivalent to its complex conjugate . The extension is then given by :-

$T(\underset{\sim}{\Gamma}(\underset{\sim}{u}').\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I}).\underset{\sim}{y}_1.\exp \underset{\sim}{\Gamma}(\underset{\sim}{h}').\underset{\sim}{u}'^{-1})$, where $T(\underset{\sim}{y}_1)(\rho_{11}+\rho_{21}) = \rho_{11}^*+\rho_{21}$

or by $T(\underset{\sim}{\Gamma}(\underset{\sim}{u}').\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I}).\underset{\sim}{y}_2.).K_{ext}.T(\exp \underset{\sim}{\Gamma}(\underset{\sim}{h}').\underset{\sim}{u}'^{-1})$, where $T(\underset{\sim}{y}_2)(\rho_{11}+\rho_{21}) = \rho_{11}+\rho_{21}^*$, this latter possibility only applying when $\tilde{\mathcal{L}} = A_\ell$ or E_6 . The two possibilities will be examined separately.

i) <u>$S'_{ext} = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}').\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I}).\underset{\sim}{y}_2).K_{ext}.T(\exp \underset{\sim}{\Gamma}(\underset{\sim}{h}').\ \underset{\sim}{u}'^{-1})$</u>

If S is outer the embedding condition becomes

$$S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}').\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I}).\underset{\sim}{y}_2).K_{ext}.T(\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}').\underset{\sim}{c}.\underset{\sim}{u}'^{-1}) \tag{2.9}$$

which ,using equation (16) in [11], reduces to a similar form to equation (25) in [11], with $\underset{\sim}{y}$ and $\underset{\sim}{h}'$ replaced by $\underset{\sim}{y}_2$ and $\underset{\sim}{h}_2'$($\underset{\sim}{h} = \underset{\sim}{h}_1' \oplus \underset{\sim}{h}_2'$) By considering the explicit forms of $\underset{\sim}{y}_2$, $\underset{\sim}{c}$,exp $\underset{\sim}{\Gamma}(\underset{\sim}{h}')$ and $\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I})$, the argument follows as in [11] section 3(c) , with $\underset{\sim}{y}$ replaced by $\underset{\sim}{y}_2$ .The embedding conditions are then A,B and C of 11 section 3(c) , with $\underset{\sim}{y}$ everywhere replaced by $\underset{\sim}{y}_2$. the proceedure for embedding is :

1) Choose an $\underset{\sim}{h}'$ such that $T(\underset{\sim}{d}_1' \otimes \underset{\sim}{I}).K_1.\exp(\mathrm{ad}\ \underset{\sim}{h}')$ generates $\mathcal{L}_1' \oplus \mathcal{L}_2'$ from $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$.

2) Choose a represenlation Γ of $\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c}$ corresponding to a set of conjugate embeddings in $\mathcal{L}_c$ and construct the set of chief centralizer elements $\underset{\sim}{g}$ (as givenfor explicit forms of Γ in [8]).

3) Put $\eta = 1$ and see if conditions A, B and C can be simultaneous satisfied for any $\underset{\sim}{g}$. Repeat with $\eta = -1$ if applicable.

4) If the conditions of 3) are satisfied, $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ is a subalgebra of $\mathcal{L}$, where $\mathcal{L}$ is generated from $\mathcal{L}_c$ by $T(\underset{\sim}{\Gamma}(\underset{\sim}{d}_1 \otimes \underset{\sim}{I})\underset{\sim}{y}_2) . K_{ext} . \exp \mathrm{ad}\, \underset{\sim}{\Gamma}(\underset{\sim}{h}'))$. (In fact the generators of $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ are the same linear combination of the generators of $\mathcal{L}$ as the generators of $\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c}$ are of those of $\mathcal{L}_c$.) If the conditions of 3) are not satisfied then $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ is not isomorphic to any subalgebra of $\mathcal{L}$ with any representation conjugate to Γ.

5) Repeat stages 2), 3) and 4) with all other non-conjugate representations (if any exist).

ii) $S'_{ext} = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}') . \underset{\sim}{\Gamma}(\underset{\sim}{d}'_1 \otimes \underset{\sim}{I}) . \underset{\sim}{y}_1 . \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}') . \underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1})$.

For the case when $\mathcal{L} = A_\ell$ or E_6 and S outer, the embedding condition reduces to the same form as in (2.9), with $\underset{\sim}{y}_2$ replaced by $\underset{\sim}{y}'_1$, where $\underset{\sim}{y}'_1 = \underset{\sim}{c}^{-1} \underset{\sim}{y}_1 \underset{\sim}{c}$, and hence the argument follows through as in the above case.

If S is inner the embedding condition becomes

$S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}') . \underset{\sim}{\Gamma}(\underset{\sim}{d}'_1 \otimes \underset{\sim}{I})\, \underset{\sim}{y}_1 . \underset{\sim}{c} . \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}') . \underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1})$,

which simplifies to a similar form to equation (40) in [11], $\underset{\sim}{d}'$ being replaced by $\underset{\sim}{d}'_1 \otimes \underset{\sim}{I}$, $\underset{\sim}{B}^{-1}\underset{\sim}{y}\,\underset{\sim}{B}$ by $\underset{\sim}{y}_1$ and $\underset{\sim}{c}$ by $\underset{\sim}{c}^{-1}$. The argument then proceeds as in [11] section 5(b), with $SL(\ell'+1, R)$ replaced by $SL(\ell'+1, R) + \mathcal{L}'_2$ and $Q_{\frac{1}{2}(\ell'+1)}$ by $Q_{\frac{1}{2}(\ell'+1)} + \mathcal{L}'_2$.

For embeddings corresponding to $\mathcal{L} = D_\ell$, and S outer, modifications as in [11] section 6(c) must be made to the above argument, as $\det \underset{\sim}{y}_1 = -1$.

c) S'₁ and S'₂ both outer, $\mathcal{L}'_1 = D_{\ell'}$, $\mathcal{L}'_2 = D_{\ell''}$.

S' is of the form $T(\underset{\sim}{u}'(\underset{\sim}{Q}'_1 \otimes \underset{\sim}{Q}'_2) \exp \underset{\sim}{h}' . \underset{\sim}{u}'^{-1}) = T(\underset{\sim}{u}' \underset{\sim}{Q} \exp \underset{\sim}{h}' . \underset{\sim}{u}'^{-1})$, where $\underset{\sim}{Q} = \underset{\sim}{Q}'_1 \otimes \underset{\sim}{Q}'_2$, and the argument follows that for the simple case given in [12] section 5(b), for embeddings corresponding to $\mathcal{L} = B_\ell$ or C_ℓ, and section 7(c) and 7(d) for embeddings corresponding to $\tilde{\mathcal{L}} = A_\ell$.

d) S_1' outer , S_2' inner , $\tilde{\mathcal{L}}_1' = D_\ell$,

S' is of the form $T(\underset{\sim}{u}'(\underset{\sim}{Q}_1' \otimes \underset{\sim}{I})\circ\exp \underset{\sim}{h}'\circ\underset{\sim}{u}'^{-1}) = T(\underset{\sim}{u}'\circ\underset{\sim}{Q}\circ\exp \underset{\sim}{h}'\circ\underset{\sim}{u}'^{-1})$, where $\underset{\sim}{Q} = \underset{\sim}{Q}_1' \otimes \underset{\sim}{I}$, and the argument follows as in c) .

e) S_1' and S_1' both outer, $\tilde{\mathcal{L}}_1' = A_\ell$ or E_6 and $\tilde{\mathcal{L}}_2' = D_\ell$.

S' is of the form $T(\underset{\sim}{u}'(\underset{\sim}{d}_1' \otimes \underset{\sim}{Q}_2')).K_1\circ\exp(\mathrm{ad}\,\underset{\sim}{h}')\circ T(\underset{\sim}{u}'^{-1})$. It can easily be shown that all possible extensions of S' can be written as either $T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I})\underset{\sim}{y}_1\circ\underset{\sim}{q}\circ\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{u}'^{-1}))$ or as $T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\Gamma(\underset{\sim}{d}_1' \otimes \underset{\sim}{I})\circ\underset{\sim}{y}_1\circ\underset{\sim}{s}).K_{ext}.T(\exp \underset{\sim}{\Gamma}(\underset{\sim}{h}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{u}'^{-1}))$, where

$T(\underset{\sim}{y}_1)(\rho_{11}(\mathcal{L}_1') + \rho_{21}(\mathcal{L}_2')) = \rho_{11}^*(\mathcal{L}_1') + \rho_{21}(\mathcal{L}_2')$,

$T(\underset{\sim}{q})(\rho_{11}(\mathcal{L}_1') + \rho_{21}(\mathcal{L}_2')) = \rho_{11}(\mathcal{L}_1') + \rho_{21}(T(\underset{\sim}{Q}_2')\mathcal{L}_2')$ and

$T(\underset{\sim}{s})(\rho_{11}(\mathcal{L}_1') + \rho_{21}(\mathcal{L}_2')) = \rho_{11}(\mathcal{L}_1') + \rho_{21}^*(T(\underset{\sim}{Q}_2')\mathcal{L}_2')$ ..

The two cases will be considered separately .

i) $S_{ext}' = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I})\circ\underset{\sim}{y}_1\cdot\underset{\sim}{q}\circ\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{u}'^{-1}))$.

If S is inner the embedding condition becomes $S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{d}_1' \otimes \underset{\sim}{I})\underset{\sim}{y}_1\circ\underset{\sim}{q}\circ\underset{\sim}{c}\circ\exp \underset{\sim}{\Gamma}(\underset{\sim}{h}')\circ\underset{\sim}{\Gamma}(\underset{\sim}{u}'^{-1}))$, and involutiveness implies

$$(\underset{\sim}{y}_1\circ\underset{\sim}{q}\circ\underset{\sim}{c})^2 = \begin{cases} \eta\,\exp(-2\underset{\sim}{\Gamma}(\underset{\sim}{h}_2')) & \text{for } SL(\ell'+1,R)\oplus\mathcal{L}_2' \\ \eta\,\exp(-2\Gamma(\underset{\sim}{h}_2')\circ\underset{\sim}{\Gamma}(-\underset{\sim}{I}(n)) & \text{for } Q_{\frac{1}{2}(\ell'+1)}\oplus\mathcal{L}_2' \end{cases} \qquad (2.10)$$

where $n = (\ell'+1)(\ell''+1)$, $\underset{\sim}{h}_2' \in 0 \oplus \tilde{\mathcal{L}}_2'$ and $\eta = \pm 1$.

For the existance of $\underset{\sim}{y}_1$ and $\underset{\sim}{q}$, the irreducible representations Γ^{jk} (where $\Gamma^{jk}(\underset{\sim}{g}_1' \otimes \underset{\sim}{g}_2') = \Gamma^j(\underset{\sim}{g}_1') \otimes \Gamma^k(\underset{\sim}{g}_2')$) must be such that either each Γ^j is eqivalent to its complex conjugate or Γ^{jk} and $\Gamma^{j*k}(=\Gamma^{j*}\otimes\Gamma^k)$ appear in pairs , and either each Γ^k is self-conjugate or Γ^{jk} and $\Gamma^{j\bar{k}}$ appear in pairs ($\underset{\sim}{\Gamma}^k(T(\underset{\sim}{Q}_2')\,\underset{\sim}{g}_2') = \underset{\sim}{\Gamma}^{\bar{k}}(\underset{\sim}{g}_2')$) , as defined in [12] and [5]) .

First taking $\mathcal{L}_1'$ as $SL(\ell'+1,R)$, ther are four types of irreducible representation appearing in the reduction of Γ, each is treated separately.

Γ^k self-conjugate and Γ^j equivalent to Γ^{j*}.

This implies the existence of $\underset{\sim}{y}_{jk}$ and $\underset{\sim}{q}_{jk}$ as in [11] such that $T(\underset{\sim}{y}_{jk})(\Gamma^{jk}) = \Gamma^{j\,k}$ and $T(\underset{\sim}{q}_{jk})(\Gamma^{jk}) = \Gamma^{jk}$, and $\underset{\sim}{y}_{jk}$ and $\underset{\sim}{q}_{jk}$ commute with $\underset{\sim}{c}'_{jk} \otimes \underset{\sim}{I}(n_j.n_k) = \underset{\sim}{c}_{jk}$, $\underset{\sim}{q}_{jk}$ and $\underset{\sim}{y}_{jk}$ can be chosen such that $\underset{\sim}{q}^2_{jk}=I$ and $\underset{\sim}{y}^2_{jk} = \epsilon_j \underset{\sim}{I}$. Since the $\underset{\sim}{y}_{jk}$ acts only on Γ^j and the $\underset{\sim}{q}_{jk}$ acts only on Γ^k , they can be chosen to be of the form $\underset{\sim}{y}_{jk}= \underset{\sim}{y}'_j \otimes \underset{\sim}{I}(n_k)$ and $\underset{\sim}{q}_{jk}= \underset{\sim}{I}(n_j) \otimes \underset{\sim}{q}'_k$, which commute . The argument follows as in [10] section 5(b) ,case (1) and [12] section 3(c) , resulting in the condition

A: If Γ^k is self conjugate and Γ^j and Γ^{j*} equivalent then

$$\exp 2\underset{\sim}{g}''_{jk} = \epsilon_j \underset{\sim}{I}(p_{jk}) .$$

Γ^k not self-conjugate and Γ^j and Γ^j equivalent .

The Γ^{jk} and $\Gamma^{j\bar{k}}$ must occur in pairs , so the part of $\underset{\sim}{q}$ corresponding to $(\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{\Gamma}^{jk}) \oplus (\underset{\sim}{I}(\underset{\sim}{p}_{jk}) \otimes \underset{\sim}{\Gamma}^{jk})$ is of the form $\underset{\sim}{q}_{jk} = \underset{\sim}{L}(2n_j.n_k.p_{jk})$ and the corresponding part of $\underset{\sim}{y}_1$ is $\underset{\sim}{y}_{jk} = (\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{y}'_j \otimes \underset{\sim}{I}(n_k)) \oplus (\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{y}'_j \otimes \underset{\sim}{I}(n_k))$. Equation (2.10) then implies $\underset{\sim}{c}'_k.\underset{\sim}{c}'_{\bar{k}} = \eta\, \epsilon_j \underset{\sim}{I}(n_j.n_k.p_{jk})$ and thus $\exp \underset{\sim}{g}''_k = \eta\, \epsilon_j \exp(-\underset{\sim}{g}''_{\bar{k}})$, which gives the condition :

B: if Γ^k is not self-conjugate and Γ^j and Γ^{j*} are equivalent, then
$p_{jk} = p_{jk}$ and $\exp \underset{\sim}{g}''_k = \eta\, \epsilon_j \exp \underset{\sim}{g}''_{\bar{k}}$.

Γ^k self-conjugate and Γ^j and Γ^{j*} not equivalent .

The Γ^{jk} and Γ^{j*k} must occur the same number of times in the reduction of Γ , the part of $\underset{\sim}{y}_1$ corresponding to $(\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{\Gamma}^{jk}) \oplus (\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{\Gamma}^{j*k})$ is of the form $\underset{\sim}{y} = \underset{\sim}{L}(2n_j.n_k.p_{jk})$. The corresponding part of $\underset{\sim}{q}$ is then of the form $(\underset{\sim}{I}(p_{jk}.n_j) \otimes \underset{\sim}{q}'_j) \oplus (\underset{\sim}{I}(p_{jk}.n_j) \otimes \underset{\sim}{q}'_j)$. the argument follows that of [11] 5(b) (2) and the following condition applies :

C: If Γ^k is self-conjugate and Γ^j is not equivalent to Γ^j , then p_{jk} must be even and equal to p_{j*k} , and and $\underset{\sim}{g}''_{jk}$ satisfies
$-\exp \underset{\sim}{g}''_{jk} = \underset{\sim}{L}(p_{jk})^{-1} \exp \underset{\sim}{g}''_{jk} \underset{\sim}{L}(p_{jk})$.

<u>Γ^k not self-conjugate and Γ^j and Γ^{j*} not equivalent</u> .

Γ^{jk} , $\Gamma^{j\bar{k}}$, Γ^{j*k} and $\Gamma^{j*\bar{k}}$ must all occur the same number of times in the reduction of Γ . The part of $\underset{\sim}{y}_1$ corresponding to $(\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{\Gamma}^{jk}) \oplus (\underset{\sim}{I}(p_{jk}) \otimes \underset{\sim}{\Gamma}^{j*k}) \oplus (\underset{\sim}{I}(p_{jk} \otimes \underset{\sim}{\Gamma}^{j\bar{k}}) \oplus (\underset{\sim}{I}(p_{jk}) \otimes \Gamma^{j*\bar{k}})$ is $\underset{\sim}{L}(2p_{jk}.n_j.n_k) \oplus \underset{\sim}{L}(2p_{jk}.n_j.n_k)$, and the corresponding part of $\underset{\sim}{q}$ is $\underset{\sim}{L}(4p_{jk}.n_j.n_k)$. Equation (2.10) then implies $\underset{\sim}{c}'_{j*k}.\underset{\sim}{c}'_{j\bar{k}} = \eta \underset{\sim}{I}(p_{jk})$ and $\underset{\sim}{c}''_{j*\bar{k}}.\underset{\sim}{c}''_{jk} = \underset{\sim}{I}(p_{jk})$ and so $\exp \underset{\sim}{g}''_{j*k} = \eta \exp(-\underset{\sim}{g}'_{j\bar{k}})$ and $\exp \underset{\sim}{g}''_{j*\bar{k}} = \eta \exp \underset{\sim}{g}''_{jk}$, which gives the condition

D : If Γ^k is not self-conjugate and Γ^j and Γ^{j*} are not equivalent, then $\exp \underset{\sim}{g}''_{j*k} = \eta \exp(-\underset{\sim}{g}'_{j\bar{k}})$ and $\exp \underset{\sim}{g}''_{j*\bar{k}} = \eta \exp(\underset{\sim}{g}''_{jk})$

Combining all four cases , the contribution to $\underset{\sim}{y}_1.\underset{\sim}{q}.\underset{\sim}{c}$ from each irreducible representation can be found as in the cases considered in [11] 5(b) and [12] 3(c) . Taking $\mathcal{L}'_1$ to be $Q_{\frac{1}{2}(\ell'+1)}$, $\underset{\sim}{\Gamma}^{jk}(-\underset{\sim}{I}(\ell'+1)(\ell''+1)) = \tau_{jk} \underset{\sim}{I}(n_j.n_k)$, the argument is essentially the same and conditions A,B,C and D apply with η everywhere replaced by $\eta\,\tau_{jk}$, where $\exp(-2\underset{\sim}{\Gamma}(\underset{\sim}{h}')) = \tau_{jk}\ \underset{\sim}{I}(n_j.n_k)$. The condition on the trace is the same for $\mathcal{L}'_1 = SL(\ell'+1,R)$ or $Q_{\frac{1}{2}(\ell'+1)}$,namely

E: Trace $(\ T(\underset{\sim}{\Gamma}(\underset{\sim}{d}'_1 \otimes \underset{\sim}{I})\underset{\sim}{y}_1.\underset{\sim}{q}.\underset{\sim}{c}.\exp(\underset{\sim}{\Gamma}(\underset{\sim}{h}'))) = -\delta$.

For an embedding corresponding to $\tilde{\mathcal{L}} = D_\ell$, where $T(y_1 . q)$ is an outer automorphism , the argument follows through as above with c everywhere replaced by $P . c$ and g' and g'' everywhere replaced by G' and G'' (as defined in [12] section 2) . The proceedure for embedding is then :

1) Choose h' such that $T(d_1' \otimes Q_2).K_1.\exp(\mathrm{ad}\, h')$ generates $\mathcal{L}_1' \oplus \mathcal{L}_2'$ from $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$.

2) Choose a representation Γ of $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$ corresponding to a set of conjugate embeddings of $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$ in $\mathcal{L}_c$, and then construct the set of chief centralizer elements g (Explicit forms for Γ and the corresponding g are given in Cornwell[9]).

3) Put $\eta = +1$, work through the set of g and see if there exists one satisfying conditions A,B,C,D and E simultaneously for $\mathcal{L}_1' = SL(\ell'+1,R)$ and det $y_1 = +1$; or their modifications for $\mathcal{L}_1' = Q_{\frac{1}{2}(\ell'+1)}$ or $\hat{\mathcal{L}}_1' = D_{\ell}$ & det $y_1 = -1$. Repeat for $\eta = -1$.

4) If condition 3) is satisfied , then $\mathcal{L}_1' \oplus \mathcal{L}_2'$ is a subalgebra of $\mathcal{L}$, the generators of $\mathcal{L}_1' \oplus \mathcal{L}_2' = \sqrt{\hat{S}_1' . \hat{S}_2'}\ (\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}')$ being the same linear combination of $\mathcal{L} = \sqrt{\hat{S}_{1ext}' . \hat{S}_{2ext}'}\ \mathcal{L}_c$ as $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$ are of $\mathcal{L}_c$.

If the conditions of 3) can not be simultaneously satisfied , then $\mathcal{L}_1' \oplus \mathcal{L}_2'$ is not isomorphic to any subalgebra of $\mathcal{L}$ with a representation conjugate to Γ .

5) Repeat stages 2) , 3) and 4) with the other non-conjugate embeddings of $\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}'$ in $\mathcal{L}_c$, if any exist .

ii) $\underline{S_{ext}' = T(\Gamma(u')\Gamma(d_1' \otimes I)\ y_1 . s).K_{ext}.T(\exp \Gamma(h').\Gamma(u')^{-1})}$

This case obviously only applies for $\tilde{\mathcal{L}} = A_\ell$ or E_6 . The case of S inner is equivalent to i) . If S is outer , ρ_{11} and ρ_{11}^* must be equivalent and ρ_{21} must be self-conjugate. The involutive condition reduces to $(y_1 . s . c)(y_1 . s . c)^* = \eta I$, and following similar arguments to those given in i) , and those given in [11] section 3(c) and [11], section 7(b) , the following conditions apply for $\mathcal{L}_1' =$ $SL(\ell'+1,R)$:

A: If Γ^j and Γ^{j*} are equivalent and Γ^k is self-conjugate , then p_{jk} is even and $\exp 2\underset{\sim}{g}'_{jk} = \epsilon_j \eta\, \underset{\sim}{I}(\, p_{jk}.n_j.n_k)$.

B: If Γ^j and Γ^{j*} are not equivalent and Γ^k is self-conjugate, then p_{jk} and p_{j*k} must be equal and $\exp \underset{\sim}{g}'_{j*k} = \eta \exp(-\underset{\sim}{g}'_{jk})$.

C: If Γ^j and Γ^{j*} are equivalent and Γ^k is not self-conjugate, then p_{jk} and $p_{j\bar{k}}$ must be equal and $\exp g'_{j\bar{k}} = \eta\ \epsilon_j \exp(-\underset{\sim}{g}'_{jk})$.

D: If Γ^j and Γ^{j*} are not equivalent and Γ^k is not self-conjugate, then p_{jk} , $p_{j\bar{k}}$, p_{j*k} and $p_{j*\bar{k}}$ must all be equal and $\exp \underset{\sim}{g}'_{j\bar{k}} = \eta \exp(-\underset{\sim}{g}'_{j*k})$ and $\exp \underset{\sim}{g}'_{j*\bar{k}} = \eta \exp(-\underset{\sim}{g}'_{jk})$.

also the trace condition becomes

E: trace $(T(\underset{\sim}{\Gamma}(\underset{\sim}{d}'_1 \otimes \underset{\sim}{I}).\underset{\sim}{y}_1.\underset{\sim}{s}).K_{ext}.\exp(ad\underset{\sim}{\Gamma}(\underset{\sim}{h}'))) = -\delta$.

For $\mathcal{L}_1' = Q_{\frac{1}{2}(\ell'+1)}$ the above conditions are modified by replacing η everywhere by $\eta\tau_{jk}$. The procedure for embedding is as in i) .

2.6.3 <u>S' outer of type (ii)</u>

The extension of Z is given in section 2.4.3 and is either of the form $T(\underset{\sim}{w}'_{ext})$ or $T(\underset{\sim}{w}'_{ext}).K_{ext}$. In the first case ρ_{11} and ρ_{21} must be equivalent , in the second case ρ_{11} and ρ^*_{21} must be equivalent ,(this case obviously only applies if $\tilde{\mathcal{L}} = A_\ell$ or E_6).

(1) <u>ρ_{11} and ρ_{21} equivalent</u>

S' is of the form $T(\underset{\sim}{u}').Z'.T(\exp \underset{\sim}{h}'.\underset{\sim}{u}'^{-1})$, and S'_{ext} is thus $T(\Gamma(\underset{\sim}{u}')\underset{\sim}{w}'_{ext}.\exp\Gamma(\underset{\sim}{h})'.\Gamma(\underset{\sim}{u}')^{-1}))$. The embedding condition for $T(\underset{\sim}{w}'_{ext})$ inner and for S inner , or $T(\underset{\sim}{w}'_{ext})$ and S both outer ($\tilde{\mathcal{L}} = D_\ell$) is then :

$$S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\,.\,\underset{\sim}{w}'_{ext}\,.\,\underset{\sim}{c}\,.\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')\,.\,\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}), \tag{2.11}$$

and involutiveness implies $(\underset{\sim}{w}'_{ext}\,.\,\underset{\sim}{c})^2 = \eta \exp(-2\underset{\sim}{\Gamma}(\underset{\sim}{h}'))$. Since Γ is of the form (2.4) and $\underset{\sim}{W}(n_j,n_k)\,.\,\underset{\sim}{W}(n_k\,.\,n_j) = \underset{\sim}{I}$ (for all j,k), equation (2.11) reduces to the conditions :

A: If Γ^{jj} appears in the reduction of Γ , then $\exp 2\underset{\sim}{g}'_{jj} = \eta\,\tau_{jj}\underset{\sim}{I}(p_{jj}n_j^2)$.

B: Γ^{jk} and Γ^{kj} must occur the same number of times in the reduction of Γ and $\exp \underset{\sim}{g}'_{jk} = \eta\,\tau_{jk}\exp(-\underset{\sim}{g}'_{kj})$.

C: Trace $(T(\underset{\sim}{w}'_{ext}\,.\,\underset{\sim}{c}\,.\exp\,\underset{\sim}{\Gamma}(\underset{\sim}{h}'))\,) = -\delta$.

A,B and C provide necessary and sufficient conditions for an embedding.

For an embedding corresponding to $\hat{\mathcal{L}} = D_\ell$, with either S or $T(\underset{\sim}{w}'_{ext})$ outer (but not both) , the above argument is repeated with $\underset{\sim}{c}$ replaced everywhere by $\underset{\sim}{P}\underset{\sim}{c}$ and $\underset{\sim}{g}'_{jk}$ replaced by $\underset{\sim}{G}'_{jk}$,(c.f. [12]section 3(b)).

For S outer and $\tilde{\mathcal{L}} = A_\ell$ or E_6 , the embedding condition becomes

$$S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\,\underset{\sim}{w}'_{ext}\,.\exp\,\underset{\sim}{\Gamma}(\underset{\sim}{h}')\,.\,\underset{\sim}{c}\,.\,\underset{\sim}{y})\,.K.T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}) \quad , \tag{2.12}$$

and involutiveness implies $(\underset{\sim}{w}'_{ext}\,.\,\underset{\sim}{c}\,.\,\underset{\sim}{y})(\underset{\sim}{w}'_{ext}\,.\,\underset{\sim}{c}\,.\,\underset{\sim}{y})^{*} = \eta\underset{\sim}{I}$. $\underset{\sim}{y}_{jk}$ can be chosen such that $\underset{\sim}{y}_{jk} = \underset{\sim}{y}_j \otimes \underset{\sim}{y}_k$. $\underset{\sim}{w}'_{ext}$ and $\underset{\sim}{c}\,.\,\underset{\sim}{y}$ then commute, and following arguments similar to those in [11] 3(c) and those for S inner (above) , equation (2.12) becomes equivalent to the following conditions:

A' : If Γ^{jk} and Γ^{j*k*} are equivalent ,then τ_{jk} must be real,and if $\eta\,\epsilon_{jk}\tau_{jk} = -1$ then p_{jk} must be even and $\underset{\sim}{g}''_{jk}$ must satisfy $-\exp \underset{\sim}{g}''_{jk} = \underset{\sim}{L}(p_{jk})^{-1}\exp \underset{\sim}{g}''_{jk}\,.\,\underset{\sim}{L}(p_{jk})$.

B': If Γ^{jk} and Γ^{j*k*} are not equivalent then, $p_{jk} = p_{j*k*}$ and $\exp \underset{\sim}{g}'_{j*k*} = \eta\,\tau_{jk}^{*}\exp(-\underset{\sim}{g}'_{jk})$.

The trace condition becomes

C: Trace ($T(\underset{\sim}{w}'_{ext}\,.\exp\,\underset{\sim}{\Gamma}(\underset{\sim}{h}')\,.\,\underset{\sim}{c}\,.\,\underset{\sim}{y})\,.K) = -\delta$.

(2) ρ_{11} and ρ_{21}^{*} equivalent. (for $\tilde{\mathcal{L}} = A_{\ell}$ or E_6 only)

S'_{ext} is given by $T(\underset{\sim}{\Gamma}(\underset{\sim}{u}').\underset{\sim}{w}'_{ext}).K_{ext}.T(\underset{\sim}{c}.\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}').\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1})$. The embedding condition (1.2) for S outer becomes

$$S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}').\underset{\sim}{w}'_{ext}).K_{ext}.T(\underset{\sim}{c}.\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}').\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}) \qquad (2.13)$$

and involutiveness implies $(\underset{\sim}{w}'_{ext}.\underset{\sim}{c})(\underset{\sim}{w}'_{ext}.\underset{\sim}{c})^{*} = \eta\, \underset{\sim}{I}$, which reduces to $\underset{\sim}{c}.\underset{\sim}{c}^{*} = \eta\, \underset{\sim}{I}$, since $\underset{\sim}{w}'_{ext}$ commutes with $\underset{\sim}{c}$ and is unitary. Thus equation (2.13) is equivalent to the conditions :

A'': If Γ^{jk} and Γ^{j*k*} are not equivalent then $p_{jk} = p_{j*k*}$.

B'': If $\eta = -1$ then p_{jk} is even and $\underset{\sim}{g}''_{jk}$ satisfies

$$-\exp \underset{\sim}{g}''_{jk} = \underset{\sim}{L}(p_{jk})^{-1} \exp \underset{\sim}{g}''_{jk} \,.\, \underset{\sim}{L}(p_{jk}) \,.$$

also the trace condition becomes

C'': Trace $(T(\underset{\sim}{w}'_{ext}.\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')^{*}.\,\hat{\underset{\sim}{c}}).K_{ext} = -\delta$.

If S is inner, then the embedding condition is given by

$$S = T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\underset{\sim}{w}'_{ext}).K_{ext}.T(\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}').\underset{\sim}{c}.\underset{\sim}{y}.\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}).K_{ext} \qquad (2.14)$$
$$= T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')\underset{\sim}{w}'_{ext}, \exp\, \underset{\sim}{\Gamma}(\underset{\sim}{h}')^{*}.\underset{\sim}{c}^{*}.\underset{\sim}{y}^{*}.\underset{\sim}{\Gamma}(\underset{\sim}{u}')^{-1}) \,.$$

and involutiveness implies $(\underset{\sim}{w}'_{ext}.\underset{\sim}{c}^{*}.\underset{\sim}{y}^{*})^2 = \eta\, \underset{\sim}{I}$.

$\underset{\sim}{w}'_{ext}$ and $\underset{\sim}{y}$ can be chosen to be real and such that $\underset{\sim}{y}_{jk}\underset{\sim}{y}_{jk} = \epsilon_{jk}\underset{\sim}{I}$. Hence equation (2.14) becomes equivalent to the conditions :

A''' : If Γ^{jk} and Γ^{j*k*} are equivalent, then if $\eta\,\epsilon_{jk} = -1$, then p_{jk} is even and $-\exp\underset{\sim}{g}''_{jk} = \underset{\sim}{L}(p_{jk})^{-1}.\exp\, \underset{\sim}{g}''_{jk}.\underset{\sim}{L}(p_{jk})$.

B''' : If Γ^{jk} and Γ^{j*k*} are not equivalent, then $p_{jk} = p_{j*k*}$ and

$$\exp\, \underset{\sim}{g}'_{j*k*} = \epsilon_{jk} \exp\, \underset{\sim}{g}'_{jk} \,.$$

The trace condition is given by

C: Trace $(T(\underset{\sim}{w}'_{ext}.\underset{\sim}{c}.\underset{\sim}{y}.\exp\underset{\sim}{\Gamma}(\underset{\sim}{h}')) = -\delta$.

The condition A,B and C : or A',B' and C': or A",B" and C" : or A‴, B‴ and C‴ (as appropriate) are necessary and sufficient conditions for the embedding condition (1.2) to be satisfied in each case (i.e. given these conditions are fulfilled a $\underset{\sim}{u}'$ exists such that equations (2.11),(2.12),(2.13) and (2.14) are satisfied). Thus the procedure for embedding is :

1) Choose an embedding representation Γ corresponding to a set of conjugate embeddings of $\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c}$ in $\mathcal{L}_c$ ($\mathcal{L}'_{1c}$ and $\mathcal{L}'_{2c}$ being isomorphic) of the form $\underset{\sim}{\Gamma}(\underset{\sim}{a}' \oplus \underset{\sim}{b}') = \rho_{11}(\underset{\sim}{a}') + \rho_{21}(\underset{\sim}{b}')$, where ρ_{11} is equivalent to ρ_{21} or ρ^*_{21} (the latter alternative applying only if $\tilde{\mathcal{L}} = A_\ell$ or E_6).
2) Put $\eta = 1$, and work through the set of $\underset{\sim}{g}$ and $\underset{\sim}{h}'$($Z'.\underset{\sim}{h}'=\underset{\sim}{h}'$) to find if the set of necessary and sufficient conditions (A,B and C; or A',B' and C' : or A",B" and C": or A‴,B‴ and C‴) can be simultaneously satisfied. Repeat with $\eta = -1$.
3) If the conditions in 2) are satisfied , then $\sqrt{Z'}(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c})$ is a subalgebra of $\mathcal{L} = \sqrt{S}\,\mathcal{L}_c = \sqrt{Z'_{ext}.Y}\,\mathcal{L}_c$, and the generators of $\sqrt{Z'}(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c})$ are the same linear combination of $\sqrt{Z'_{ext}.Y}\,\mathcal{L}_c$ as those of $\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c}$ are of $\mathcal{L}_c$.
   If the conditions of 2) are not satisfied then $\sqrt{Z'}(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c})$ is not isomorphic to any subalgebra of $\mathcal{L}$ with an embedding conjugate to Γ .
4) Repeat with all other non-conjugate embeddings (if any exist).

2.6.4 Example : Embeddings of real forms of $A_1 \oplus A_1$ in real forms of A_3.

The involutive automorphisms of $A_1 \oplus A_1$ are either inner or of type (ii) and so sections 2.6.1 and 2.6.3 apply . There are two non-conjugate embeddings of $SU(2) \otimes SU(2)$ in $SU(4)$ which are :

I) $\Gamma_{2,1} \oplus \Gamma_{1,2}$: $\Gamma(\underset{\sim}{g}_1' \otimes \underset{\sim}{g}_2') = (\underset{\sim}{\Gamma}_2(\underset{\sim}{g}_1') \otimes \underset{\sim}{\Gamma}_1(\underset{\sim}{g}_2')) \oplus (\underset{\sim}{\Gamma}_1(\underset{\sim}{g}_1') \otimes \underset{\sim}{\Gamma}_2(\underset{\sim}{g}_2'))$,

which can be realised in the form $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1^{(1)\prime}) = \underset{\sim}{h}_1$, $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1^{(2)\prime}) = \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1^{(1)\prime}) = \underset{\sim}{e}_1$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1^{(2)\prime}) = \underset{\sim}{e}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{f}_1^{(1)\prime}) = \underset{\sim}{f}_1$ and $\underset{\sim}{\Gamma}(\underset{\sim}{f}_1^{(2)\prime}) = \underset{\sim}{f}_3$.

II) $\Gamma_{2,2}$: $\underset{\sim}{\Gamma}(\underset{\sim}{g}_1' \otimes \underset{\sim}{g}_2') = \underset{\sim}{\Gamma}_2(\underset{\sim}{g}_1') \otimes \underset{\sim}{\Gamma}_2(\underset{\sim}{g}_2')$,

which can be realised in the form $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1^{(1)\prime}) = \underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 + \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1^{(2)\prime}) = \underset{\sim}{h}_1 + \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1^{(1)\prime}) = \underset{\sim}{e}_{1,2} + \underset{\sim}{e}_{1,3}$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1^{(2)\prime}) = \underset{\sim}{e}_1 + \underset{\sim}{e}_3$ ..etc.

Representation I is reducible and representation II is irreducible. They will be dealt with separately .

Representation I

The centralizer consists of elements of the form $c_1\underset{\sim}{I}(2) \oplus c_2\underset{\sim}{I}(2)$, since $\Gamma = \Gamma_{2,1} \oplus \Gamma_{1,2}$. As $\underset{\sim}{c} \in SU(4)$, $c_1 = c_2^{-1} = e^{i\psi}$ for some $\psi \in [0,2\pi)$. Hence the chief centralizer generator $\hat{g}$ is $i\psi(\underset{\sim}{I}(2) \oplus (-\underset{\sim}{I}(2)) = -i\psi(\underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 + \underset{\sim}{h}_3)$.

a) S' and S both inner automorphisms.

The embedding conditions A and B become

A: trace $(\exp(\mathrm{ad}(\underset{\sim}{\Gamma}(\underset{\sim}{h}') - i\psi(\underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 + \underset{\sim}{h}_3)\,) = -\delta$,

B: $\exp(\mathrm{ad}(\underset{\sim}{\Gamma}(\underset{\sim}{h}') - i\psi(\underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 + \underset{\sim}{h}_3)$ is involutive ,

which are satisfied for the cases given in table 2.3.

b) S' inner and S outer

$\underset{\sim}{y}$ can be taken as $\underset{\sim}{y} = \underset{\sim}{y}_{21} \oplus \underset{\sim}{y}_{12}$, where $\underset{\sim}{y}_{21} = \underset{\sim}{y}_{12} = i\underset{\sim}{L}(2)$, which gives $\epsilon_{12} = \epsilon_{21} = 1$. τ_{jk} is +1 except when $\underset{\sim}{h}' = \pi\underset{\sim}{h}_1^{(1)\prime}$,when τ_{12} is -1 , and when $\underset{\sim}{h}' = \pi\underset{\sim}{h}_1^{(2)\prime}$, when τ_{21} is -1 . In these two cases condition B can not be satisfied , so there are no embeddings of $SU(2) \otimes SU(1,1)$ and $SU(1,1) \otimes SU(2)$ in Q_2 or $SL(4,R)$ with this representation . All embedding conditions are satisfied for embedding $SU(2) \otimes SU(2)$ in Q_2 and $SU(1,1) \otimes SU(1,1)$ in $SL(4,R)$.

Table 2.3 : Embedding of real forms of $A_1 \oplus A_1$ in real forms of A_3 with representation I and S' inner

		$SU(2) \otimes SU(2)$	$SU(1,1) \otimes SU(2)$	$SU(1,1) \otimes SU(2)$	$SU(1,1) \otimes SU(1,1)$
	$\underset{\sim}{h}'$	0	$\frac{1}{2}\pi i \underset{\sim}{h}_1^{(1)'}$	$\frac{1}{2}\pi i \underset{\sim}{h}_1^{(2)'}$	$\frac{1}{2}i\pi(\underset{\sim}{h}_1^{(1)'} + \underset{\sim}{h}_1^{(2)'})$
	$\underset{\sim}{\Gamma}(\underset{\sim}{h}')$	0	$\frac{1}{2}\pi i \underset{\sim}{h}_1$	$\frac{1}{2}\pi i \underset{\sim}{h}_3$	$\frac{1}{2}\pi i(\underset{\sim}{h}_1 + \underset{\sim}{h}_3)$
S inner	Values of ψ satisfying B	$0, \frac{\pi}{2}$	$\frac{\pi}{4}, \frac{3\pi}{4}$	$\frac{\pi}{4}, \frac{3\pi}{4}$	$0, \frac{\pi}{2}$
	Trace in A	15, -1	3	3	-1
	Embeddings exist in :	SU(4) SU(2,2)	SU(3,1)	SU(3,1)	SU(2,2)
S outer	values of ψ satisfying B	$0, \frac{\pi}{2}$	none	none	$0, \frac{\pi}{2}$
	Trace in C	5	-	-	-3
	Embeddings exist in :	Q_2	-	-	SL(4,R)

c) <u>S' of type (ii) S inner</u>

Z' can be extended to $T(\underset{\sim}{w}'_{ext})$, where $\underset{\sim}{w}'_{ext} = \underset{\sim}{L}(4)$ and S' is of the form $Z' \circ \exp(\mathrm{ad}\, \underset{\sim}{h}')$, where $Z'\underset{\sim}{h}' = \underset{\sim}{h}'$, and hence $\underset{\sim}{h}' = \gamma i(\underset{\sim}{h}_1^{(1)} + \underset{\sim}{h}_1^{(2)})$. Involutiveness makes $\gamma = p\pi$ or $(p+\frac{1}{2})\pi$, (p being an integer). Thus $\tau_{21} = \tau_{12} = +1$ for $\gamma = p\pi$ and -1 for $\gamma = (p+\frac{1}{2})\pi$. If $\eta = 1$ then condition B becomes $e^{i\psi}\underset{\sim}{I}(2) = \tau_{21}\, e^{-i\psi}\underset{\sim}{I}(2)$, which is only satisfied if $\gamma = p\pi$ and $\psi = 0$; if $\eta = -1$ then B becomes $e^{i\psi}\underset{\sim}{I}(2) = -\tau_{21}e^{-i\psi}\underset{\sim}{I}(2)$, which is only satisfied if $\gamma = (p+\frac{1}{2})$ and $\psi = 0$. In both cases the trace of $T(\underset{\sim}{w}'_{ext} \circ \underset{\sim}{c} \circ \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}'))$ is -1, so condition C implies that $SL(2,C)$ can be embedded in $SU(2,2)$ with generators $i(\underset{\sim}{h}_1 + \underset{\sim}{h}_3)$, $(\underset{\sim}{h}_3 - \underset{\sim}{h}_1)$, $(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 + \underset{\sim}{f}_1 + \underset{\sim}{f}_3)$, $i(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$, $i(\underset{\sim}{e}_1 - \underset{\sim}{e}_3 + \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$, $(\underset{\sim}{e}_3 - \underset{\sim}{e}_1 + \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$.

d) <u>S' of type (ii) and S outer.</u>

Since $\Gamma_{1,2}$ and $\Gamma_{2,1}$ are both equivalent to their complex conjugates, condition A' applies. If $\eta = 1$, the condition is only satisfied if $\gamma = p\pi$ (since $\epsilon_{12} = \epsilon_{21} = 1$ and $p_{12} = p_{21} = 1$), and the trace of $T(\underset{\sim}{w}'_{ext} \circ \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}') \circ \underset{\sim}{c} \circ \underset{\sim}{y} \circ) K_{ext}$ is 5. Thus $SL(2,C)$ can be embedded in Q_2 with generators $i(\underset{\sim}{h}_1 + \underset{\sim}{h}_3)$, $\underset{\sim}{h}_1 - \underset{\sim}{h}_3$, $\underset{\sim}{e}_1 + \underset{\sim}{e}_3 + \underset{\sim}{f}_1 + \underset{\sim}{f}_3$, $i(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$, $i(\underset{\sim}{e}_1 - \underset{\sim}{e}_3 + \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$, $\underset{\sim}{e}_3 - \underset{\sim}{e}_1 + \underset{\sim}{f}_1 - \underset{\sim}{f}_3$. If $\eta = -1$ condition A' is satisfied if $\gamma = (p+\frac{1}{2})\pi$ and the trace of $T(\underset{\sim}{w}'_{ext} \circ \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}') \circ \underset{\sim}{c} \circ \underset{\sim}{y}) \circ K_{ext}$ is -3. Thus $SL(2,C)$ can be embedded in $SL(4,R)$ with generators $\underset{\sim}{h}_1 + \underset{\sim}{h}_3$, $i(\underset{\sim}{h}_1 - \underset{\sim}{h}_3)$, $i(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 + \underset{\sim}{f}_1 + \underset{\sim}{f}_3)$, $(\underset{\sim}{e}_1 + \underset{\sim}{e}_3 - \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$, $\underset{\sim}{e}_1 - \underset{\sim}{e}_3 + \underset{\sim}{f}_1 - \underset{\sim}{f}_3$, $i(\underset{\sim}{e}_3 - \underset{\sim}{e}_1 + \underset{\sim}{f}_1 - \underset{\sim}{f}_3)$.

<u>Representation II</u>

Since this an irreducible representation the centralizer elements are a constant times the identity and hence $T(\underset{\sim}{c}) = T(\underset{\sim}{I})$ for all centralizer elements $\underset{\sim}{c}$.

a) S' and S both inner

exp(ad $\Gamma(h')$) is involutive if exp(ad h') is involutive. Condition B is thus satisfied and condition A implies that $SU(2)\otimes SU(2)$ can be embedded in $SU(4)$ (when $h'=0$ the trace in condition A is 15) : similarly $SU(2)\otimes SU(1,1)$, $SU(1,1)\otimes SU(2)$ and $SU(1,1)\otimes SU(1,1)$ can all be embedded in $SU(2,2)$, (the trace in A being -1) .

b) S' inner and S outer

y can be taken as $y_2 \otimes y_2$ where $y_2 = iL(2)$ so

$$y = \begin{pmatrix} 0, & 0, & 0, & -1 \\ 0, & 0, & -1, & 0 \\ 0, & -1, & 0, & 0 \\ -1, & 0, & 0, & 0 \end{pmatrix} .$$

Conditions A,B and C are satisfied almost trivially ,the trace in C being 5 for $h'=0$ and $\frac{i\pi}{2}(h_1^{(1)'}+h_1^{(2)'})$ and -3 for $h'=\frac{i\pi}{2}h_1^{(1)'}$ and $\frac{i\pi}{2}h_1^{(2)'}$. Thus $SU(2)\otimes SU(2)$ and $SU(1,1)\otimes SU(1,1)$ can be embedded in Q_2 , and $SU(2)\otimes SU(1,1)$ and $SU(1,1)\otimes SU(2)$ can be embedded in $SL(4,R)$ with this representation .

c) S' of type (ii) and S inner

S'_{ext} is of the form $T(w'_{ext}\cdot \exp\Gamma(h'))$, where w'_{ext} as given in 2.4.2 is

$$w'_{ext} = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} .$$

h' must be of the form $i\gamma\,(h_1^{(1)'}+h_1^{(2)'})$ with $\gamma = p\pi$ or $(p+\frac{1}{2})\pi$,p being an integer. Condition A applies and is satisfied for $\eta = 1$, the trace in C is 3 so $SL(2,C)$ can be embedded in $SU(3,1)$ with this representation .

d) S' of type (ii) S outer

Using the form of w'_{ext} and y given above , conditions A',B' and C' are satisfied for an embedding of $SL(2,C)$ in $SL(4,R)$,(the permitted values of γ give a trace of +1 or -3 ,but no real form of A_3 generated by an S which is outer, has character -1).The embeddings of this section are summarised in table 2.4.

Table 2.4 : The embeddings of real forms of $A_1 \oplus A_1$ in real forms of A_3 (embedding (2))

Embeddings exist with the representations, whose numbers are indicated.

Real forms of A_3	Real forms of $A_1 \oplus A_1$				
	$SU(2) \otimes SU(2)$	$SU(1,1) \otimes SU(2)$	$SU(2) \otimes SU(1,1)$	$SU(1,1) \otimes SU(1,1)$	$SL(2,\mathbb{C})$
$SU(4)$	I,II	none	none	none	none
$SU(3,1)$	none	I	I	none	II
$SU(2,2)$	I	II	II	I,II	I
Q_2	I,II	none	none	II	I
$SL(4,R)$	none	II	II	I	I,II

2.7 Embedding 3

Since all the ρ_{ij} are non-zero Γ can be expressed in the form $\Gamma(\underset{\sim}{a}'\oplus\underset{\sim}{b}')=(\rho_{11}(\underset{\sim}{a}')+\rho_{21}(\underset{\sim}{b}'))\oplus(\rho_{12}(\underset{\sim}{a}')+\rho_{22}(\underset{\sim}{b}'))$, or in the form $(\underset{\sim}{B}_1\oplus\underset{\sim}{B}_2)((\rho^c_{11}(\underset{\sim}{a}')+\rho^c_{21}(\underset{\sim}{b}'))\oplus(\rho^c_{12}(\underset{\sim}{a}')+\rho^c_{22}(\underset{\sim}{b}')))(\underset{\sim}{B}_1\oplus\underset{\sim}{B}_2)^{-1}$, where the ρ^c_{ij} are in canonical form and completely reduced (and ρ^c_{1i} and ρ^c_{2i} commute). The extension of an automorphism of type (i) $S'=\hat{S}'_1.\hat{S}'_2$, can be taken as $S'_{ext}=(\hat{S}'_{1ext1}.\hat{S}'_{2ext1}).(\hat{S}'_{1ext2}.\hat{S}'_{2ext2})=\hat{S}'_{ext1}.\hat{S}'_{ext2}$, which is also of type (i). The extension of a type (ii) automorphism $S'=Z'.\hat{S}'_1.\hat{S}'_2$ is the product of the extension of Z' and the extension of the type (i) automorphism $S'_1.S'_2$.

If Z'_{ext} is of type (i) then it is of the form $\hat{Z}'_{ext1}.\hat{Z}'_{ext2}$, where Z'_{exti} is the extension of Z' in $\mathcal{L}_i$ and is either $T(\underset{\sim}{w}'_{exti})$ (ρ_{1i} and ρ_{2i} equivalent) or $T(\underset{\sim}{w}'_{exti}).K_{exti}$ (ρ_{1i} and ρ^*_{2i} equivalent), as shown in section 2.4.2 . If Z'_{ext} is of type (ii) then it is of the form $Z'_{ext}=Z.\hat{Z}'_1.\hat{Z}'_2$, where the Z'_i are given by $Z'_1=T(\underset{\sim}{B}_2.\underset{\sim}{B}_1^{-1})$ and $Z'_2=T(\underset{\sim}{B}_1.\underset{\sim}{B}_2^{-1})$ for $\rho^c_{11}=\rho^c_{22}$ and $\rho^c_{12}=\rho^c_{21}$, and by $Z'_1=T(\underset{\sim}{B}_2).K.T(\underset{\sim}{B}_1^{-1})$ and $Z'_2=T(\underset{\sim}{B}_1).K.T(\underset{\sim}{B}_2^{-1})$ for $\rho^c_{11}=\rho^{c*}_{22}$ and $\rho^c_{12}=\rho^{c*}_{21}$ as shown in section 2.4.2.

The automorphism Y may also be of type (i) or type (ii) . If of type (i) , it can be written as $\hat{Y}_1.\hat{Y}_2$, where the Y_i can be expressed as $T(\underset{\sim}{B}_i).Y^c_i.T(\underset{\sim}{B}_i)^{-1}$. If Y is of type (ii) it can be written as $Z.\hat{Y}^{(1)}.\hat{Y}^{(2)}$, where $Y^{(1)}(\rho_{11}(a')+\rho_{21}(b'))=\rho_{12}(a')+\rho_{22}(b')$ and $Y^{(2)}(\rho_{12}(a')+\rho_{22}(b'))=\rho_{11}(a')+\rho_{21}(b')$, implying that ρ_{j1} and ρ_{j2} are equivalent (j=1,2) or ρ_{j1} and ρ^*_{j2} are equivalent (this latter alternative only applies when $\tilde{\mathcal{L}}_1=\hat{\mathcal{L}}_2=A_\ell$ or E_6) .
If S'_{ext} is of type(i) there is no restriction on the ρ_{ij} which prevents the $\underset{\sim}{B}_i$ being chosen such that $\rho^c_{j1}=\rho^c_{j2}$ (or ρ^{c*}_{j2}) . The $Y^{(i)}$ can thus be chosen as $Y^{(1)}=T(\underset{\sim}{B}_2).Y^c_1.T(\underset{\sim}{B}_1^{-1})$ and $Y^{(2)}=T(\underset{\sim}{B}_1).Y^c_2.T(\underset{\sim}{B}_2^{-1})$,

in the first case, and $Y^{(1)} = T(\underset{\sim}{B}_2).K.Y_1^c.T(\underset{\sim}{B}_1^{-1})$ and $Y^{(2)} = T(\underset{\sim}{B}_1).Y_2^c.K.T(\underset{\sim}{B}_2^{-1})$, in the second case, the Y_i^c are as defined in section 2.6. If S'_{ext} is of type (ii), then there is a restriction on the ρ_{ij}, preventing the same choice of $\underset{\sim}{B}_i$ as above, namely $\rho_{11}^c = \rho_{22}^c$ (or ρ_{22}^{c*}) and $\rho_{12}^c = \rho_{21}^c$ (or ρ_{21}^{c*}). The $\underset{\sim}{B}_i$ may however be chosen such that $\rho_{j1}^c = \sum_k(\Gamma^k \otimes I(p_k))$ and $\rho_{j2}^c = \sum_k(I(p_k)\otimes\Gamma^k)$, or if $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$, ρ_{j2}^c may be replaced by ρ_{j2}^{c*}. The $Y^{(i)}$ can then be written as $Y^{(1)} = T(\underset{\sim}{B}_2.\underset{\sim}{B}_1^{-1}.\underset{\sim}{w}'_{ext1}).Y_1$ and $Y^{(2)} = T(\underset{\sim}{B}_1.\underset{\sim}{B}_2^{-1}.\underset{\sim}{w}'_{ext2})Y_2$ for ρ_{j1} and ρ_{j2} equivalent, or $Y^{(1)} = T(\underset{\sim}{B}_2).K.T(\underset{\sim}{B}_1^{-1}.\underset{\sim}{w}'_{ext1}).Y_1$ and $Y^{(2)} = T(\underset{\sim}{B}_1).K.T(\underset{\sim}{B}_2^{-1}.\underset{\sim}{w}'_{ext2}).Y_2$ for ρ_{j1} and ρ_{j2}^* equivalent, where the Y_i are those given for Y of type (i) above.

There are four cases to discuss, namely those for which S'_{ext} and S are both of type(i), both of type (ii) or of different types.

a) <u>S'_{ext} and S both of type (i)</u>

If S is given by $S = \widehat{S}_1.\widehat{S}_2$, then the embedding condition (1.2) becomes $\widehat{S}_1.\widehat{S}_2 = \widehat{S}'_{ext1}.\widehat{S}'_{ext2}.\widehat{Y}_1.\widehat{Y}_2$, which reduces to $S_i = S'_{exti}.Y_i$, (i=1,2), which is the necessary and sufficient condition for embedding $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ in $\mathcal{L}_i$, and so the problem reduces to that considered in section 2.6.

Thus a necessary and sufficient condition for $\sqrt{S'}(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c})$ to be a subalgebra of $\mathcal{L}_1 \oplus \mathcal{L}_2$ with embedding representation $\rho_1 \oplus \rho_2$ is that $\sqrt{S'}(\mathcal{L}'_{1c} \oplus \mathcal{L}'_{2c})$ be a subalgebra of both $\mathcal{L}_1$ and $\mathcal{L}_2$ with embedding representation ρ_1 and ρ_2 respectively.

b) <u>S'_{ext} of type (i) and S of type (ii)</u> (implying that $\tilde{\mathcal{L}}_1 \equiv \tilde{\mathcal{L}}_2$)

If S is given by $S = Z.\widehat{S}_1.\widehat{S}_2$, the embedding condition (1.2) is

$$Z.\widehat{S}_1.\widehat{S}_2 = \widehat{S}'_{ext1}.\widehat{S}'_{ext2}.Z.\widehat{Y}^{(1)}.\widehat{Y}^{(2)} \qquad (2.15)$$

The trace of both sides is zero and involutiveness implies

$$(S'_{ext1}.Y^{(2)}) = (S'_{ext2}.Y^{(1)})^{-1} . \qquad (2.16)$$

If $\rho^c_{i1} = \rho^c_{i2}$, S'_{exti} may be taken to be $T(\underset{\sim}{B}_i).S^c_i.T(\underset{\sim}{B}_i^{-1})$, where S^c_i is the extension of S' into $\mathcal{L}_i$ for an embedding representation $\rho^c_{1i} + \rho^c_{2i}$. In this case $Y^c_1 = Y^c_2 = Y^c$ and $S^c_1 = S^c_2 = S^c$. If $\rho^c_{i1} = \rho^{c*}_{i2}$ then $Y^c_1 = K.Y^c_2.K = Y^c$ and $S^c_1 = K.S^c_2.K = S^c$. In both cases (2.16) reduces to

$$(S^c.Y^c)^2 = I . \qquad (2.17)$$

Since $S^c.Y^c$ is the extension of an involutive automorphism , Y^c can always be chosen such that (2.17) is satisfied. Since $S_1 = S_2^{-1}$ is comparatively arbitrary , the embedding condition (2.15) can always be satisfied for a suitable choice of S_1 , if and only if $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ can be embedded in some real form of $\tilde{\mathcal{L}}_1$.

Hence a necessary and sufficient condition for $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ to be a subalgebra of $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$ with embedding representation $\rho_1 \oplus \rho_2$ is that ρ_1 and ρ_2 , or ρ_1 and ρ^*_2 (for $\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 = A_\ell$ or E_6) be equivalent , and that <u>some</u> real form of $\tilde{\mathcal{L}}_1$ contains $\mathcal{L}'_1 \oplus \mathcal{L}'_2$ as a subalgebra with embedding representation ρ_1 .

c) <u>S'_{ext} of type (i) and S of type (ii)</u> ($\tilde{\mathcal{L}}'_1 \equiv \tilde{\mathcal{L}}'_2$ and $\tilde{\mathcal{L}}_1 \equiv \tilde{\mathcal{L}}_2$)

The embedding condition (1.2) becomes for $S' = \hat{S}'_1.\hat{S}'_2$

$\hat{S}_1.\hat{S}_2 = Z.\hat{Z}_1.\hat{Z}_2.\hat{S}'_{ext1}.\hat{S}'_{ext2}.Z.\hat{Y}^{(1)}.\hat{Y}^{(2)}$, which is equivalent to

$$\left.\begin{aligned} S_1 &= Z'_2.\ S'_{ext2}.Y^{(1)} \\ S_2 &= Z'_1.\ S'_{ext1}.Y^{(2)} \end{aligned}\right\} \qquad (2.18)$$

S'_{exti} can be expressed in the form $T(\underset{\sim}{B}_i).S^c_i.T(\underset{\sim}{B}_i)^{-1}$. Substituting in (2.18) one obtains the equations $S_i = Z'_{exti}.S'_{exti}.Y_i$, (and after further manipulation), which are the necessary and sufficient conditions in all cases for embedding $\sqrt{Z}'(\mathcal{L}'_{1c}\oplus\mathcal{L}'_{2c})$ in $\mathcal{L}_1$ and $\mathcal{L}_2$ separately, as in a). This case is in fact equivalent to a), since if a Y of type (ii) exists, then Z' could have been extended to an automorphism of type (i).

d) <u>S'_{ext} and S both of type (ii)</u> ($\tilde{\mathcal{L}}'_1 \equiv \tilde{\mathcal{L}}'_2$ and $\tilde{\mathcal{L}}_1 \equiv \tilde{\mathcal{L}}_2$)

Y is of type (i) and so the embedding condition (1.2) becomes

$$Z.\hat{S}_1.\hat{S}_2 = Z.\hat{Z}'_1.\hat{Z}'_2.\hat{S}'_{ext1}.\hat{S}'_{ext2}.\hat{Y}_1.\hat{Y}_2 .$$

The trace of both sides is zero, and involutiveness implies

$$(Z'_1.S'_{ext1}.Y_1) = (Z'_2.S'_{ext2}.Y_2)^{-1}$$

For both the allowable forms of representation ($\rho^c_{11}=\rho^c_{22}$ and $\rho^c_{12}=\rho^c_{21}$, and $\rho^c_{11}=\rho^{c*}_{22}$ and $\rho^c_{12}=\rho^{c*}_{21}$) this reduces to

$$(S^c.Y^c)^2 = I . \qquad (2.19)$$

Since the choice of $S'=\hat{S}'_1.\hat{S}'_2$ is relatively arbitrary, (2.19) can always be satisfied by a suitable choice of S' and Y^c. (e.g. S' and Y^c both the identity), provided some real form of $\tilde{\mathcal{L}}'_1\oplus\tilde{\mathcal{L}}'_2$ is a subalgebra of some real form of $\mathcal{L}_1$. All choices of S' satisfying (2.19) provide conjugate embeddings.

Hence a necessary and sufficient condition for embedding $\sqrt{Z}'(\mathcal{L}'_{1c}\oplus\mathcal{L}'_{2c})$ in $\sqrt{Z}(\mathcal{L}_{1c}\oplus\mathcal{L}_{2c})$ is that $\mathcal{L}'_{1c}\oplus\mathcal{L}'_{2c}$ is a subalgebra of $\mathcal{L}_{1c}$, with embedding representation $\rho^c_{11}+\rho^c_{21}$. Possible representations are conjugate to, either $(\rho^c_{11}+\rho^c_{21})\oplus(\rho^c_{11}+\rho^c_{21})$ or (in the case of $\tilde{\mathcal{L}}_1 = A$ or E_6) $(\rho^c_{11}+\rho^c_{21})+(\rho^{c*}_{11}+\rho^{c*}_{21})$.

2.7.2. <u>Example</u> : The embedding of real forms of $A_1\oplus A_1$ in real forms of $A_3\oplus A_3$.

There are two non-conjugate embeddings of $A_1\oplus A_1$ in A_3 of type 2, $\Gamma_{2,1}\oplus\Gamma_{1,2}$ and $\Gamma_{2,2}$. Hence there are three non-conjugate embeddings od $A_1 \oplus A_1$ in $A_3 \oplus A_3$ with embeddings of type 3, which are :

I) $(\Gamma_{2,1} \oplus \Gamma_{1,2}) \oplus (\Gamma_{2,1} \oplus \Gamma_{1,2})$

II) $\Gamma_{2,2} \oplus \Gamma_{2,2}$

III) $(\Gamma_{2,1} \oplus \Gamma_{1,2}) \oplus \Gamma_{2,2}$

Using the resultts from section 2.6.4 for the embedding of $A_1 \oplus A_1$ in A_3 , table 2.5 gives all possible embeddings of real forms of $A_1 \oplus A_1$ in real forms of $A_3 \oplus A_3$ of type 3 .

Table2.5 : Possible embeddings of real forms of $A_1 \oplus A_1$ in those of $A_3 \oplus A_3$ (embedding 3)

Embeddings exist with the representations indicated.

Real forms of $A_3 \oplus A_3$	Real forms of $A_1 \oplus A_1$: $SU(2) \otimes SU(2)$	$SU(1,1) \otimes SU(2)$ $SU(2) \otimes SU(1,1)$	$SU(1,1) \otimes SU(1,1)$	$SL(2,C)$
$SU(4) \otimes SU(4)$	I,II,III	none	none	none
$SU(4) \otimes SU(3,1)$	none	none	none	none
$SU(4) \otimes SU(2,2)$	I	none	none	none
$SU(4) \otimes Q_2$	I,II,III	none	none	none
$SU(4) \otimes SL(4,R)$	none	none	none	none
$SU(3,1) \otimes SU(3,1)$	none	I	none	II
$SU(3,1) \otimes SU(2,2)$	none	III	none	III
$SU(3,1) \otimes Q_2$	none	none	none	III
$SU(3,1) \otimes SL(4,R)$	none	III	none	II,III
$SU(2,2) \otimes SU(2,2)$	I	II	I,II,III	I
$SU(2,2) \otimes Q_2$	I,III	none	II,III	I
$SU(2,2) \otimes SL(4,R)$	none	II	I,III	I
$Q_2 \otimes Q_2$	I,II,III	none	II	I
$Q_2 \otimes SL(4,R)$	none	none	III	I,III
$SL(4,R) \otimes SL(4,R)$	none	II	I	I,II,III
$SL(4,C)$	I,II	I,II	I,II	I,II

2.8 Embedding 4

2.8.1 Basic theory

In this case $\rho_{22} = 0$ and all the other ρ_{ij} are non-zero $(i,j = 1,2)$. The argument follows that in section 2.7 exactly, but with $\rho_{22}=0$. The automorphism Z' has no extension, neither can any real form of $\tilde{\mathcal{L}}_1' \oplus \tilde{\mathcal{L}}_2'$ be embedded in $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$. Thus S'_{ext} and S are both of type (i) and section 2.7 (a) applies .

Hence a necessary and sufficient condition for embedding $\mathcal{L}_1' \oplus \mathcal{L}_2'$ in $\mathcal{L}_1 \oplus \mathcal{L}_2$ with representation $(\rho_{11} + \rho_{21}) \oplus \rho_{12}$ is that $\mathcal{L}_1' \oplus \mathcal{L}_2'$ is a subalgebra of $\mathcal{L}_1$ with the embedding representation $\rho_{11} + \rho_{21}$ and that $\mathcal{L}_1'$ is a subalgebra of $\mathcal{L}_2$ with the embedding representation ρ_{12} . There are no embeddings of $\sqrt{Z'}(\mathcal{L}_{1c}' \oplus \mathcal{L}_{2c}')$ and no embeddings into $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$.

2.8.2 <u>Example</u>: The embedding of real forms of $A_1 \oplus A_1$ in real forms of $A_3 \oplus A_2$.

Since there are two non-conjugate embeddings of $A_1 \oplus A_1$ in A_3 , there are four non-conjugate embeddings of $A_1 \oplus A_1$ in $A_3 \oplus A_2$ of type 4 , namely :

I) $\Gamma_{2,2} \oplus (\Gamma_2 \oplus \Gamma_1)$,

II) $\Gamma_{2,2} \oplus \Gamma_3$,

III) $(\Gamma_{2,1} \oplus \Gamma_{1,2}) \oplus (\Gamma_2 \oplus \Gamma_1)$,

IV) $(\Gamma_{2,1} \oplus \Gamma_{1,2}) \oplus \Gamma_3$.

Using results from section 2.6.4 and the results from 10 and 11 for the embedding of A_1 in A_2 (quoted in section 2.3.2) , all possible embeddings can be obtained and are given in table 2.6.

Table 26 : The embeddings of real forms of $A_1 \oplus A_1$ in those of $A_3 \oplus A_2$ (embedding 4)

Embeddings exist with the representations whose numbers are indicated.

Real forms of $A_3 \oplus A_2$	Real forms of $A_1 \oplus A_1$				
	$SU(2) \otimes SU(2)$	$SU(2) \otimes SU(1,1)$	$SU(1,1) \otimes SU(2)$	$SU(1,1) \otimes SU(1,1)$	$SL(2,C)$
$SU(4) \otimes SU(3)$	I,II,III,IV	none	none	none	none
$SU(4) \otimes SU(2,1)$	I,III	none	none	none	none
$SU(4) \otimes SL(3,R)$	II,IV	none	none	none	none
$SU(3,1) \otimes SU(3)$	none	none	none	III,IV	none
$SU(3,1) \otimes SU(2,1)$	none	III	III,IV	none	none
$SU(3,1) \otimes SL(3,R)$	none	IV	IV,III	none	none
$SU(2,2) \otimes SU(3)$	III,IV	I,II	none	none	none
$SU(2,2) \otimes SU(2,1)$	III	I	I,II	I,II,III,IV	none
$SU(2,2) \otimes SL(3,R)$	IV	I	I,II	I,II,III,IV	none
$SL(4,R) \otimes SU(3)$	none	I,II	none	none	none
$SL(4,R) \otimes SU(2,1)$	none	I	I,II	III,IV	none
$SL(4,R) \otimes SL(3,R)$	none	II	I,II	III,IV	none
$Q_2 \otimes SU(3)$	I,II,III,IV	none	none	none	none
$Q_2 \otimes SU(2,1)$	I,III	none	none	I,II	none
$Q_2 \otimes SL(3,R)$	II,IV	none	none	I,II	none

2.9 Embeddings of SL(2,C)

The group SL(2,C) is physically a particularly important group as it is the universal covering group of the homogeneous Lorentz group.

Barut and Raczka [1] have classified the obvious embeddings of SL(2,C) by using certain well known isomorphisms between simple Lie algebras , as given in table 2 of [1] . They have not considered the possibility of other non-conjugate embeddings, nor in fact have they examined the form of the representation at all . They have missed several obvious embeddings , including the embedding of SL(2,C) in SU(2,2) , and have missed others by not considering embeddings other than the obvious ones . An example of this is the embedding of SL(2,C) in SU(3,1) and ND_6 . There may also be two non-conjugate embeddings of SL(2,C) in the same real form , as for example in SL(4,R) , a possibility which Barut and Raczka did not consider .

By the theory in section 2.5 , 2.6 , 2.7 and 2.8 in this chapter it is possible to list all possible embeddings of SL(2,C) in classical semi-simple Lie algebras . This process is however very time consuming and will not therefore be carried out for large Lie algebras , though the method will in these cases be indicated.

For embeddings of SL(2,C) in simple Lie algebras , the methods illustrated in section 2.6 are used . One example of embedding SL(2,C) in real forms of A_3 was given in section 2.6.4 in which it was shown that SL(2,C) can be embedded in SU(3,1) with representation II , in SU(2,2) and Q_2 with representation I , and in SL(4,R) with representations I and II . Further examples of possible embeddings of SL(2,C) in simple Lie algebras are now given .

(a) Embeddings in B_2 and C_2

As B_2 and C_2 are isomorphic , it will be sufficient to consider embeddings in real forms of B_2. There are two non-conjugate representations of $SU(2)\otimes SU(2)$ of dimension 5 , for which ρ_{11} is equivalent to ρ_{21} , namely

I) $\Gamma_{2,2} \oplus \Gamma_{1,1}$,

II) $\Gamma_{2,1} \oplus \Gamma_{1,2} \oplus \Gamma_{1,1}$

Since Γ_2 of $SU(2)$ is pseudo real $\Gamma_{2,1}$, $\Gamma_{1,2}$ and ($\Gamma_{2,1} \oplus \Gamma_{1,2} \oplus \Gamma_{1,1}$) of $SU(2)\otimes SU(2)$ will also be pseudo real . As $SO(5)$ is real ,representation II does not provide an embedding of $A_1 \oplus A_1$ in B_2 , (c.f. Cornwell [9]) . Representation I provides the obvious embeddings of $SL(2,C)$,(isomorphic to $SO(3,1)$) in $SO(3,2)$ and $SO(4,1)$,(Sp_4 and NSp_4^2) . The above analysis shows that these embeddings are unique up to conjugacy .

(b) Embeddings in D_2

There is an isomorphism between $A_1 \oplus A_1$ and D_2 given in the form of the embedding of $SU(2)\otimes SU(2)$ in $SO(4)$ with representation $\underset{\sim}{X}(\Gamma_{2,2}) \underset{\sim}{X}^{-1}$ where $\underset{\sim}{X}$ is given by

$$\underset{\sim}{X} = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & -1 \\ 0, & 1, & 0, & 0 \end{pmatrix}$$

which gives $SL(2,C)$ isomorphic to $SO(3,1)$.

(c) Embedding in A_3 and D_3

As in section 2.6.4 possible embeddings are of $SL(2,C)$ in $SU(3,1)$ and $SL(4,R)$ with representation $\Gamma_{2,2}$, in $SU(2,2)$ and Q_2 and $SL(4,R)$ with representation $\Gamma_{21} \oplus \Gamma_{1,2}$, in $SO(4,2)$, $SO(5,1)$ and $SO(3,3)$ with representation $\Gamma_{2,2} \oplus \Gamma_{1,1} \oplus \Gamma_{1,1}$, and ND_6 and $SO(3,3)$ with representation $\Gamma_{3,1} \oplus \Gamma_{1,3}$.

(d) Embedding in B_3

There are two non-conjugate embeddings of $SU(2)\otimes SU(2)$ in $SO(7)$ with ρ_{11} and ρ_{21} equivalent , namely :

I) $\Gamma_{2,2} \oplus \Gamma_{1,1} \oplus \Gamma_{1,1} \oplus \Gamma_{1,1}$

II) $\Gamma_{3,1} \oplus \Gamma_{1,3} \oplus \Gamma_{1,1}$.

Representation I is of the same form as representation I in B_2 and is thus equivalent to embedding SL(2,C) in real forms of B_2 and then into real forms of B_3 . Thus with representation I , SL(2,C) may be embedded in SO(6,1) ,SO(5,2) and SO(4,3) . Representation II can be realised in the form $\underset{\sim}{X}(\Gamma_{3,1} \oplus \Gamma_{1,3} \oplus \Gamma_{1,1}) \underset{\sim}{X}^{-1}$ where X is given by

$$\underset{\sim}{X} = \begin{pmatrix} 0, & 0, & 0, & 0, & 0, & 0, & 1 \\ 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & \frac{i}{\sqrt{2}}, & 0, & 0, & \frac{1}{\sqrt{2}}, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & \frac{i}{\sqrt{2}}, & 0, & 0, & \frac{1}{\sqrt{2}}, & 0, & 0 \end{pmatrix} ,$$

giving the explicit form of the embedding as $\Gamma(h_1^{(1)\prime}) = -(2h_1 + 2h_2 + h_3)$, $\Gamma(h_1^{(2)\prime}) = -(2h_2 + h_3)$, $\Gamma(e_1^{(1)\prime}) = -f_{1,2} + \frac{1}{2} f_{1,2,3,3}$, $\Gamma(f_1^{(1)\prime}) = -e_{1,2} + \frac{1}{2} e_{1,2,3,3}$, $\Gamma(e_1^{(2)\prime}) = f_2 + \frac{1}{2} f_{2,3,3}$, $\Gamma(f_1^{(2)\prime}) = e_2 + \frac{1}{2} e_{2,3,3}$. $Z'_{ext} = T(w'_{ext})$ where

$$w'_{ext} = \underset{\sim}{X} \left(\begin{array}{ccc|ccc|c} 0, & 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ \hline -1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & -1, & 0, & 0, & 0, & 0 \\ \hline 0, & 0, & 0, & 0, & 0, & 0, & 1 \end{array}\right) \underset{\sim}{X}^{-1} = \begin{pmatrix} 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & i, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & -1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & i \end{pmatrix}$$

The centralizer elements are of the form $\underset{\sim}{c} = \underset{\sim}{X}(c'_1 \underset{\sim}{I}(3) + c'_2 \underset{\sim}{I}(3) + c'_3 \underset{\sim}{I}(1)) \underset{\sim}{X}^{-1}$, where $c_1'^2 = c_2'^2 = c_3'^2 = 1$ (from orthogonality) and $c_1'^3 . c_2'^3 . c_3' = 1$, (from the special condition), so $c'_1 = c'_2 = \pm 1$, $c'_3 = +1$ or $c'_1 = -c'_2 = \pm 1$, $c'_3 = -1$. This gives the chief centralizer generator $\underset{\sim}{g}$ as diag($0, \pm n\pi, \pm n\pi, \pm n\pi, \pm n\pi, \pm n\pi, \pm n\pi$) $= n\pi i(h_1 + \frac{1}{2} h_3)$

for $n = 0,1$. S'_{ext} is of the form $Z'_{ext} . \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}'))$ where $\underset{\sim}{h}' = \gamma(\underset{\sim}{h}_1^{(1)'} + \underset{\sim}{h}_1^{(2)'})$. Condition B is satisfied for $n = 1$ and $\gamma = 0$, $\frac{m\pi}{2}$, which gives a trace of -3 in condition C , implying that SL(2,C) can be embedded in SO(4,3) with this representation .

Embeddings of SL(2,C) in other simple Lie algebras may be treated in a similar way . We now examine the possible embeddings of SL(2,C) in semi-simple classical Lie algebras which have a semi-simple extension of the form $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$.

Embeddings of type 1

From the results of section 2.5 , it can be seen that SL(2,C) can only be embedded in algebras of the form $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$, where there exists a real form of $\tilde{\mathcal{L}}'_i$ which can be embedded in a real form of $\hat{\mathcal{L}}_i$ (with representation ρ_i) , $i=1,2$. The condition is then only on the embedding, namely $\rho_1 \equiv \rho_2$ or ρ_2^* . Hence SL(2,C) can be embedded in the following :-

i) SL(ℓ+1,C) , $\ell \geqslant 1$, with representation conjugate to $\rho_1 \oplus \rho_1$,
ii) SO(n,C) , $n \geqslant 3$, " " " " ,
iii) Sp(n,C) $n \geqslant 1$, " " " " .

Embeddings of type 3

The embedding condition is again a condition on the embedding, $\underset{\sim}{\Gamma}(\underset{\sim}{a}' \oplus \underset{\sim}{b}')$ being in one of the following forms :

$\underset{\sim}{B}((\rho_1(\underset{\sim}{a}') + \rho_2(\underset{\sim}{b}')) \oplus (\rho_2(a') + \rho_1(\underset{\sim}{b}'))) \underset{\sim}{B}^{-1}$,

$\underset{\sim}{B}((\rho_1(\underset{\sim}{a}') + \rho_2(\underset{\sim}{b}')) \oplus (\rho_1(\underset{\sim}{a}') + \rho_2(\underset{\sim}{b}'))) \underset{\sim}{B}^{-1}$,

unless SL(2,C) can be embedded separately in both $\mathcal{L}_1$ and $\mathcal{L}_2$. These possibilities give rise to the following embeddings of SL(2,C):

i) $\mathcal{L}_1 \oplus \mathcal{L}_2$,where SL(2,C) can be embedded in both $\mathcal{L}_1$ and $\mathcal{L}_2$, separately with embeddings of type 2 .

ii) $\sqrt{Z}(\mathcal{L}_{1c} \oplus \mathcal{L}_{2c})$, where some real form of $A_1 \oplus A_1$ can be embedded in some real form of $\tilde{\mathcal{L}}_1 \equiv \tilde{\mathcal{L}}_2$ with representation ρ_1. Examples of these embeddings are in $SL(\ell+1,C)$ for $\ell \geqslant 3$, $SO(n,C)$ for $n \geqslant 4$ and $Sp(n,C)$ for $n \geqslant 2$.

The algebras in which $SL(2,C)$ may be embedded may be classified as follows:

1) Algebras with a simple complex extension , for which the methods described in section 2.6 apply , such as :
   - (a) $SO(p,q)$ for $p \geqslant 3$ and $q \geqslant 1$,
   - (b) $SU(p,q)$ for $p+q \geqslant 4$ and $q \geqslant 1$,
   - (c) $SL(n,R)$ for $n \geqslant 4$,
   - (d) Q_n for $n \geqslant 2$
   - (e) NSp_{2q}^{2p} for $p \geqslant 1$ and $q \geqslant 2$,
   - (f) ND_{2n} for $n \geqslant 3$,
   - (g) Sp_{2n} for $n \geqslant 2$;
2) Simple Lie algebras with semi-simple complex extensions , for which the methods described in section 2.5 and 2.7 apply , including
   - (a) $SL(n,C)$ $n \geqslant 2$ (embedding 1) . If $n \geqslant 4$ there is also an embedding of type 3 .
   - (b) $SO(n,C)$ $n \geqslant 3$ (embedding 1). If $n \geqslant 6$ there is also an embedding of type 3 .
   - (c) $Sp(n,C)$ $n \geqslant 1$ (embedding 1). If $n \geqslant 2$ there is also an embedding of type 3 .
3) Real semi-simple Lie algebras $\mathcal{L}_1 \oplus \mathcal{L}_2$ where $\mathcal{L}_1$ and $\mathcal{L}_2$ are any of the algebras mentioned in 1) .
4) Algebras whose complex extensions are the direct-sum of of more than two simple Lie algebras.

It is possible to discover whether $SL(2,C)$ is a subalgebra of a given semi-simple real Lie algebra , with not more than two simple Lie algebras in their decomposition by methods given in this chapter. For algebras whose complex extension has more than two simple Lie algebras in its decomposition , the argument follows on the same lines as those in section 2.5 , 2.7 and 2.8.

It should be emphasised that there are in general several non-conjugate embeddings of $SL(2,C)$ in each of the above algebras .

CHAPTER 3

THE PROBLEM WHEN $\mathcal{L}'$ AND/OR $\mathcal{L}$ ARE EXCEPTIONAL LIE ALGEBRAS

3.1 Introduction

This chapter is concerned with the case when at least one of the real semi-simple Lie algebras is non-classical . The case when both algebras are simple is studied in detail and the generalisation for the non-simple cases indicated .

In section 3.2 explicit forms(matrix representations) are obtained for the defining representation of all the exceptional Lie algebras . Patera [31 , 32] has done this previously for G_2 and F_4, but his method involves embedding the exceptional Lie algebra in a classical Lie algebra . The method used here involves finding the weights of the defining representation , and is easier to extend to E_6 ,E_7 and E_8 . Mehta [29] and Mehta and Srivastava [30] give some useful information , including the fundamental weights of all exceptional Lie algebras .

Section 3.3 examines the inner involutive automorphisms of all the exceptional Lie algebras and the real forms generated by them . Section 3.4 examines the outer involutive automorphisms of E_6 (the only exceptional Lie algebra which has outer automorphisms) and the real forms generated by them.

The methods and conditions of [10 , 11 and 12] are modified and extended to the cases where $\mathcal{L}'$ and/or $\mathcal{L}$ are exceptional in section 3.5 . Section 3.6 contains examples , including all the subalgebras of G_2 and all embeddings of SL(2,C) in F_4 and E_6 . These latter are new results .

Hitherto the only application of the exceptional simple Lie algebras in elementary particle physics has been the use of the compact real form of G_2 as an internal symmetry group in place of the more commonly used SU(3) group (c.f. for example Behrends, Landovitz and Tunkelang [2], Behrends and Sirlin [3], and Behrends [4].) The major obstacle has been that the embedding relationships of the various exceptional real Lie algebras with each other and with the classical real Lie algebras has never previously been studied. This present work removes this obstacle, and, as mentioned above, shows that several of the simple exceptional real Lie algebras contain the physically important Lie algebra of the quantum-mechanical homogeneous Lorentz group SL(2,C) as a subalgebra.

3.2. Explicit forms for Complex Exceptional Lie Algebras

The Dynkin diagram and Cartan matrix $\underset{\sim}{A}$ of every exceptional Lie algebra are given in the appendix 1 The general method used here to obtain an explicit form (matrix representation) uses the weights of the defining representation. The fundamental dominant weights $\lambda_i(\underset{\sim}{h})$, $i = 1,\ldots\ell$, (ℓ is the rank of the algebra) must satisfy $\lambda_i(\underset{\sim}{h}_j) = \delta_{ij}$, $j = 1,\ldots\ell$, where the $\underset{\sim}{h}_j$ ($j = 1,\ldots,\ell$) are a basis for the Cartan subalgebra in canonical form (c.f. [24] pages 121,225). This implies that $\lambda_i(\underset{\sim}{h}) = (\underset{\sim}{A}^{-1})_{ki}\ \alpha_k(h)$ (the summation convention is always implied unless otherwise stated), where the α_k, $k = 1,\ldots\ell$ are the simple roots. Every irreducible representation is characterized by its highest weight $M = m_i\lambda_i$ (written as $(m_1,\ldots,m_\ell)$) and its dimension d is given by Weyl's dimensionality formula

$$d = \Pi \frac{\sum_{i=1}^{\ell} (m_i+1)k_i w_i}{\sum_{i=1}^{\ell} w_i k_i}$$

, (c.f. [24] page 257), where the product is over all sequences $\{k_i\}$ ($i = 1,\ldots\ell$) for which $k_i\alpha_i$ is a positive root, and w_i is the weight of the root α_i in the Dynkin diagram. Then the lowest dimensional representation may be found and its weights calculated using the fact that if Λ is a weight and $\dfrac{2(\Lambda,\ \alpha_i)}{(\alpha_i,\alpha_i)} = k$ (a positive integer), then $(\Lambda - j\alpha_i)$ is also a weight for $j = 1,\ldots k$. The explicit matrix expression for the basis elements of the Cartan subalgebra are then given by $(\underset{\sim}{h}_j)_{pq} = M_p(\underset{\sim}{h}_j)\ \delta_{pq}$ (summation not implied), where M_p is the pth weight. The generators $\underset{\sim}{e}_j$ and $\underset{\sim}{f}_j$ associated with the simple roots are found from equations (1.7), namely $[\underset{\sim}{e}_i,\ \underset{\sim}{h}_j] = A_{ji}\ \underset{\sim}{e}_i$, $[\underset{\sim}{f}_i,\ \underset{\sim}{h}_j] = -A_{ji}\ \underset{\sim}{f}_i$, and $[\underset{\sim}{e}_i,\ \underset{\sim}{f}_j] = \delta_{ij}\ \underset{\sim}{h}_j$.

2.1 An explicit form for G_2

The fundamental dominant weights of G_2 are $\lambda_1 = 2\alpha_1 + 3\alpha_2$ and $\lambda_2 = \alpha_1 + 2\alpha_2$. The fundamental representations (1,0) and (0,1) have dimensions 14 and 7, respectively. The weights of the lowest dimensional representation are $\alpha_1+2\alpha_2$, $\alpha_1+\alpha_2$, α_1, 0, $-\alpha_1$, $-\alpha_1 - \alpha_2$, and $-\alpha_1 - 2\alpha_2$. Thus the generators of the Cartan subalgebra may be taken as

$$\left.\begin{aligned} \underset{\sim}{h}_1 &= \text{diagonal } (0,\ 1,\ -1,\ 0,\ -1,\ 1,\ 0)\ , \\ \underset{\sim}{h}_2 &= \text{diagonal } (+1,\ -1,\ 2,\ 0,\ -2,\ +1,\ -1)\ , \end{aligned}\right\} \qquad (3.1)$$

and the generators associated with simple roots may be taken as

$$\left.\begin{aligned} \underset{\sim}{e}_1 &= (\underset{\sim}{I}_{3,2} + \underset{\sim}{I}_{6,5})\ , \\ \underset{\sim}{e}_2 &= (\underset{\sim}{I}_{2,1} + \sqrt{2}\ \underset{\sim}{I}_{4,3} + \sqrt{2}\ \underset{\sim}{I}_{5,4} + \underset{\sim}{I}_{7,6})\ , \end{aligned}\right\} \qquad (3.2)$$

and $\underset{\sim}{f}_i = -\tilde{\underset{\sim}{e}}_i$ $(i = 1,2)$, where $\underset{\sim}{I}_{jk}$ is the seven dimensional matrix given by $(\underset{\sim}{I}_{ij})_{pq} = \delta_{ip}\ \delta_{jq}$. The generators $\underset{\sim}{e}_{a,b,c}\ldots$ associated with non-simple roots $\alpha_a+\alpha_b+\alpha_c\ldots$ are calculated from $\underset{\sim}{e}_{ij} = [\underset{\sim}{e}_i,\ \underset{\sim}{e}_j]$. They are

$$\left.\begin{aligned} \underset{\sim}{e}_{12} &= (+\underset{\sim}{I}_{3,1} - \sqrt{2}\ \underset{\sim}{I}_{4,2} + \sqrt{2}\ \underset{\sim}{I}_{6,4} - \underset{\sim}{I}_{7,5})\ , \\ \underset{\sim}{e}_{122} &= 2(-\sqrt{2}\ \underset{\sim}{I}_{4,1} + \underset{\sim}{I}_{5,2} + \underset{\sim}{I}_{6,3} - \sqrt{2}\ \underset{\sim}{I}_{7,4})\ , \\ \underset{\sim}{e}_{1222} &= 6(+\underset{\sim}{I}_{5,1} - \underset{\sim}{I}_{7,3})\ , \\ \underset{\sim}{e}_{11222} &= 6(\underset{\sim}{I}_{6,1} + \underset{\sim}{I}_{7,2})\ . \end{aligned}\right\} \qquad (3.3)$$

3.2.2 An explicit form for F_4

The fundamental dominant weights of F_4 are $\lambda_1 = 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$, $\lambda_2 = 3\alpha_1+6\alpha_2+8\alpha_3+4\alpha_4$, $\lambda_3 = 2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4$, $\lambda_4 = \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$, and the dimensions of the fundamental representations (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1) are 52, 1274, 273 and 26 respectively. The regular representation is thus (1,0,0,0) and the lowest dimensional representation (0,0,0,1), a matrix representation of which is given below.

The generators of the Cartan subalgebra are

$$\left.\begin{aligned} \underset{\sim}{h}_1 &= \text{diagonal } (0,0,0,+1,+1,-1,+1,-1,-1,0,0,0,0,0,0,0,0,+1,+1,-1,+1,-1,-1,0,0,0), \\ \underset{\sim}{h}_2 &= \text{diagonal } (0,0,+1,-1,0,0,0,+1,+1,-1,-1,0,0,0,0,+1,+1,-1,-1,0,0,0,+1,-1,0,0), \\ \underset{\sim}{h}_3 &= \text{diagonal } (0,+1,-1,+1,-1,+1,0,-1,0,+1,+2,-1,0,0,+1,-2,-1,0,+1,0,-1,+1,-1,+1,-1,0), \\ \underset{\sim}{h}_4 &= \text{diagonal } (+1,-1,0,0,+1,0,-1,+1,-1,+1,-1,+2,0,0,-2,+1,-1,+1,-1,+1,0,-1,0,0,+1,-1), \end{aligned}\right\} \qquad (3.4)$$

and the generators corresponding to simple roots are

$$
\left.
\begin{aligned}
\underset{\sim}{e}_1 &= \underset{\sim}{I}_{6,4}+\underset{\sim}{I}_{8,5}+\underset{\sim}{I}_{9,7}+\underset{\sim}{I}_{20,18}+\underset{\sim}{I}_{22,19}+\underset{\sim}{I}_{23,21}\,,\\
\underset{\sim}{e}_2 &= \underset{\sim}{I}_{4,3}+\underset{\sim}{I}_{10,8}+\underset{\sim}{I}_{11,9}+\underset{\sim}{I}_{18,16}+\underset{\sim}{I}_{19,17}+\underset{\sim}{I}_{24,23}\,,\\
\underset{\sim}{e}_3 &= \underset{\sim}{I}_{3,2}+\underset{\sim}{I}_{5,4}+\underset{\sim}{I}_{8,6}+\underset{\sim}{I}_{12,10}+a\underset{\sim}{I}_{13,11}+b\underset{\sim}{I}_{14,11}+a\underset{\sim}{I}_{16,13}+b\underset{\sim}{I}_{16,14}+\underset{\sim}{I}_{17,15}+\underset{\sim}{I}_{21,19}+\underset{\sim}{I}_{23,22}+\underset{\sim}{I}_{25,24}\,,\\
\underset{\sim}{e}_4 &= \underset{\sim}{I}_{2,1}+\underset{\sim}{I}_{7,5}+\underset{\sim}{I}_{9,8}+\underset{\sim}{I}_{11,10}+b\underset{\sim}{I}_{13,12}+a\underset{\sim}{I}_{14,12}+b\underset{\sim}{I}_{15,13}+a\underset{\sim}{I}_{15,14}+\underset{\sim}{I}_{17,16}+\underset{\sim}{I}_{19,18}+\underset{\sim}{I}_{22,20}+\underset{\sim}{I}_{26,25}
\end{aligned}
\right\},
\tag{3.5}
$$

where $a = \frac{1}{2}(1+\sqrt{3})$ and $b = \frac{1}{2}(1-\sqrt{3})$. The generators corresponding to the non-simple roots ($\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_2+2\alpha_3$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+2\alpha_3$, $\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3$, $\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$, $\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4$, $\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$, $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$) can be obtained easily from these. Finally $\underset{\sim}{f}_i$ is given by $-\underset{\sim}{\tilde{e}}_i$.

3.2.3 An explicit form of E_6

The fundamental dominant weights of E_6 are $\lambda_1 = \frac{1}{3}(4\alpha_1+5\alpha_2+6\alpha_3+4\alpha_4+2\alpha_5+3\alpha_6)$, $\lambda_2 = \frac{1}{3}(5\alpha_1+10\alpha_2+12\alpha_3+8\alpha_4+4\alpha_5+6\alpha_6)$, $\lambda_3 = 2\alpha_1+4\alpha_2+6\alpha_3+4\alpha_4+2\alpha_5+3\alpha_6$, $\lambda_4 = \frac{1}{3}(4\alpha_1+8\alpha_2+12\alpha_3+10\alpha_4+5\alpha_5+6\alpha_6)$, $\lambda_5 = \frac{1}{3}(2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6)$ and $\lambda_6 = \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$. The dimensions of the fundamental representations are 27, 351, 2925, 351, 27 and 78 respectively. Thus the regular representation is (0,0,0,0,0,1) and the lowest dimensional representations are (1,0,0,0,0,0) and (0,0,0,0,1,0). A matrix representation of (1,0,0,0,0,0) is given below. The generators of the Cartan subalgebra are the diagonal matrices

$$
\left.
\begin{aligned}
\underset{\sim}{h}_1 &= \text{diagonal }(-1,+1,0,0,0,0,0,0,0,-1,0,0,0,-1,+1,-1,-1,+1,+1,+1,0,-1,+1,0,0,0,0),\\
\underset{\sim}{h}_2 &= \text{diagonal }(0,-1,+1,0,0,0,0,0,-1,+1,0,-1,-1,+1,0,+1,0,0,0,-1,+1,0,-1,+1,0,0,0),\\
\underset{\sim}{h}_3 &= \text{diagonal }(0,0,-1,+1,0,0,0,-1,+1,0,-1,+1,0,0,0,-1,+1,0,-1,+1,0,0,0,-1,+1,0,0),\\
\underset{\sim}{h}_4 &= \text{diagonal }(0,0,0,-1,+1,0,-1,+1,0,0,0,-1,+1,-1,0,+1,0,-1,+1,0,0,0,0,0,-1,+1,0),\\
\underset{\sim}{h}_5 &= \text{diagonal }(0,0,0,0,-1,+1,0,-1,-1,-1,+1,+1,0,+1,-1,0,0,+1,0,0,0,0,0,0,0,-1,+1),\\
\underset{\sim}{h}_6 &= \text{diagonal }(0,0,0,-1,-1,-1,+1,+1,0,0,+1,0,0,0,0,0,-1,0,0,-1,-1,+1,+1,+1,0,0,0),
\end{aligned}
\right\}
\tag{3.6}
$$

and the generators corresponding to simple roots are

$$\left.\begin{aligned}
\underset{\sim}{e}_1 &= \underset{\sim}{I}_{1,2} + \underset{\sim}{I}_{10,15} + \underset{\sim}{I}_{14,18} + \underset{\sim}{I}_{16,19} + \underset{\sim}{I}_{17,20} + \underset{\sim}{I}_{22,23} \\
\underset{\sim}{e}_2 &= \underset{\sim}{I}_{2,3} + \underset{\sim}{I}_{9,10} + \underset{\sim}{I}_{12,14} + \underset{\sim}{I}_{13,16} + \underset{\sim}{I}_{20,21} + \underset{\sim}{I}_{23,24} \\
\underset{\sim}{e}_3 &= \underset{\sim}{I}_{3,4} + \underset{\sim}{I}_{8,9} + \underset{\sim}{I}_{11,12} + \underset{\sim}{I}_{16,17} + \underset{\sim}{I}_{19,20} + \underset{\sim}{I}_{24,25} \\
\underset{\sim}{e}_4 &= \underset{\sim}{I}_{4,5} + \underset{\sim}{I}_{7,8} + \underset{\sim}{I}_{12,13} + \underset{\sim}{I}_{14,16} + \underset{\sim}{I}_{18,19} + \underset{\sim}{I}_{25,26} \\
\underset{\sim}{e}_5 &= \underset{\sim}{I}_{5,6} + \underset{\sim}{I}_{8,11} + \underset{\sim}{I}_{9,12} + \underset{\sim}{I}_{10,14} + \underset{\sim}{I}_{15,18} + \underset{\sim}{I}_{26,27} \\
\underset{\sim}{e}_6 &= \underset{\sim}{I}_{4,7} + \underset{\sim}{I}_{5,8} + \underset{\sim}{I}_{6,11} + \underset{\sim}{I}_{17,22} + \underset{\sim}{I}_{20,23} + \underset{\sim}{I}_{21,24}
\end{aligned}\right\} \quad (3.7)$$

3.2.4 An explicit form of E_7

The fundamental dominant weights are $\lambda_1 = 2\alpha_1+3\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7$, $\lambda_2 = 3\alpha_1+6\alpha_2+8\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6+4\alpha_7$, $\lambda_3 = 4\alpha_1+8\alpha_2+12\alpha_3+9\alpha_4+6\alpha_5+3\alpha_6+6\alpha_7$, $\lambda_4 = \frac{1}{2}(6\alpha_1+12\alpha_2+18\alpha_3+15\alpha_4+10\alpha_5+5\alpha_6+9\alpha_7)$, $\lambda_5 = 2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+2\alpha_6+3\alpha_7$, $\lambda_6 = \frac{1}{2}(2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+3\alpha_7)$, $\lambda_7 = 4\alpha_1+8\alpha_2+12\alpha_3+9\alpha_4+6\alpha_5+3\alpha_6+6\alpha_7$. The dimensions of the fundamental representations are 133, 8645, 365750, 27664, 1539, 56, 912. Hence the regular representation is (1,0,0,0,0,0,0) and the lowest dimensional representation is (0,0,0,0,0,1,0). A matrix representation of the latter is given. The generators of the Cartan subalgebra are the diagonal matrices

$$\left.\begin{aligned}
\underset{\sim}{h}_1 = &-\underset{\sim}{I}_{6,6}+\underset{\sim}{I}_{7,7}+\underset{\sim}{I}_{8,8}+\underset{\sim}{I}_{9,9}-\underset{\sim}{I}_{19,19}+\underset{\sim}{I}_{20,20}+\underset{\sim}{I}_{21,21}+\underset{\sim}{I}_{22,22}-\underset{\sim}{I}_{30,30}-\underset{\sim}{I}_{31,31}-\underset{\sim}{I}_{32,32}+\underset{\sim}{I}_{33,33} \\
&+\underset{\sim}{I}_{34,34}+\underset{\sim}{I}_{35,35}-\underset{\sim}{I}_{41,41}+\underset{\sim}{I}_{42,42}-\underset{\sim}{I}_{48,48}+\underset{\sim}{I}_{49,49}-\underset{\sim}{I}_{51,51}-\underset{\sim}{I}_{52,52}+\underset{\sim}{I}_{53,53}-\underset{\sim}{I}_{54,54}-\underset{\sim}{I}_{55,55} \\
&-\underset{\sim}{I}_{56,56}, \\
\underset{\sim}{h}_2 = &-\underset{\sim}{I}_{5,5}+\underset{\sim}{I}_{6,6}-\underset{\sim}{I}_{9,9}+\underset{\sim}{I}_{10,10}+\underset{\sim}{I}_{11,11}-\underset{\sim}{I}_{18,18}+\underset{\sim}{I}_{19,19}-\underset{\sim}{I}_{22,22}+\underset{\sim}{I}_{23,23}+\underset{\sim}{I}_{24,24}-\underset{\sim}{I}_{29,29}+\underset{\sim}{I}_{30,30} \\
&-\underset{\sim}{I}_{33,33}-\underset{\sim}{I}_{34,34}-\underset{\sim}{I}_{35,35}+\underset{\sim}{I}_{36,36}-\underset{\sim}{I}_{40,40}+\underset{\sim}{I}_{41,41}+\underset{\sim}{I}_{43,43}-\underset{\sim}{I}_{47,47}+\underset{\sim}{I}_{48,48}-\underset{\sim}{I}_{50,50}+\underset{\sim}{I}_{51,51} \\
&-\underset{\sim}{I}_{54,54}, \\
\underset{\sim}{h}_3 = &-\underset{\sim}{I}_{4,4}+\underset{\sim}{I}_{5,5}-\underset{\sim}{I}_{8,8}+\underset{\sim}{I}_{9,9}-\underset{\sim}{I}_{11,11}+\underset{\sim}{I}_{12,12}+\underset{\sim}{I}_{13,13}-\underset{\sim}{I}_{17,17}+\underset{\sim}{I}_{18,18}-\underset{\sim}{I}_{21,21}+\underset{\sim}{I}_{22,22}-\underset{\sim}{I}_{24,24} \\
&+\underset{\sim}{I}_{25,25}-\underset{\sim}{I}_{30,30}+\underset{\sim}{I}_{31,31}-\underset{\sim}{I}_{36,36}+\underset{\sim}{I}_{37,37}-\underset{\sim}{I}_{39,39}+\underset{\sim}{I}_{40,40}-\underset{\sim}{I}_{43,43}-\underset{\sim}{I}_{46,46}+\underset{\sim}{I}_{47,47}-\underset{\sim}{I}_{51,51} \\
&+\underset{\sim}{I}_{52,52}, \\
\underset{\sim}{h}_4 = &-\underset{\sim}{I}_{3,3}+\underset{\sim}{I}_{4,4}-\underset{\sim}{I}_{9,9}-\underset{\sim}{I}_{10,10}+\underset{\sim}{I}_{11,11}-\underset{\sim}{I}_{13,13}+\underset{\sim}{I}_{14,14}+\underset{\sim}{I}_{15,15}-\underset{\sim}{I}_{18,18}-\underset{\sim}{I}_{19,19}-\underset{\sim}{I}_{20,20}+\underset{\sim}{I}_{21,21} \\
&-\underset{\sim}{I}_{25,25}+\underset{\sim}{I}_{26,26}-\underset{\sim}{I}_{31,31}+\underset{\sim}{I}_{32,32}+\underset{\sim}{I}_{33,33}-\underset{\sim}{I}_{37,37}-\underset{\sim}{I}_{38,38}+\underset{\sim}{I}_{39,39}-\underset{\sim}{I}_{45,45}+\underset{\sim}{I}_{46,46}+\underset{\sim}{I}_{50,50} \\
&+\underset{\sim}{I}_{51,51},
\end{aligned}\right\}$$

$$
\begin{aligned}
\mathbf{h}_5 = {} & -\mathbf{I}_{2,2}+\mathbf{I}_{3,3}-\mathbf{I}_{11,11}-\mathbf{I}_{12,12}+\mathbf{I}_{13,13}-\mathbf{I}_{15,15}+\mathbf{I}_{16,16}+\mathbf{I}_{17,17}+\mathbf{I}_{18,18}+\mathbf{I}_{19,19}+\mathbf{I}_{20,20} \\
& -\mathbf{I}_{26,26}+\mathbf{I}_{27,27}-\mathbf{I}_{32,32}-\mathbf{I}_{33,33}+\mathbf{I}_{34,34}-\mathbf{I}_{39,39}-\mathbf{I}_{40,40}-\mathbf{I}_{41,41}-\mathbf{I}_{42,42}+\mathbf{I}_{43,43}-\mathbf{I}_{44,44} \\
& +\mathbf{I}_{45,45}+\mathbf{I}_{55,55}, \\
\mathbf{h}_6 = {} & -\mathbf{I}_{1,1}+\mathbf{I}_{2,2}-\mathbf{I}_{13,13}-\mathbf{I}_{14,14}+\mathbf{I}_{15,15}-\mathbf{I}_{27,27}+\mathbf{I}_{28,28}-\mathbf{I}_{34,34}+\mathbf{I}_{35,35}+\mathbf{I}_{36,36}+\mathbf{I}_{37,37} \\
& +\mathbf{I}_{38,38}+\mathbf{I}_{39,39}+\mathbf{I}_{40,40}+\mathbf{I}_{41,41}+\mathbf{I}_{42,42}-\mathbf{I}_{43,43}-\mathbf{I}_{45,45}-\mathbf{I}_{46,46}-\mathbf{I}_{47,47}-\mathbf{I}_{48,48}-\mathbf{I}_{49,49} \\
& -\mathbf{I}_{55,55}+\mathbf{I}_{56,56}, \\
\mathbf{h}_7 = {} & -\mathbf{I}_{5,5}-\mathbf{I}_{6,6}-\mathbf{I}_{7,7}+\mathbf{I}_{8,8}-\mathbf{I}_{12,12}-\mathbf{I}_{13,13}-\mathbf{I}_{14,14}-\mathbf{I}_{15,15}-\mathbf{I}_{16,16}+\mathbf{I}_{17,17}-\mathbf{I}_{22,22}-\mathbf{I}_{23,23} \\
& +\mathbf{I}_{24,24}+\mathbf{I}_{29,29}+\mathbf{I}_{30,30}-\mathbf{I}_{37,37}+\mathbf{I}_{38,38}+\mathbf{I}_{39,39}+\mathbf{I}_{44,44}+\mathbf{I}_{45,45}+\mathbf{I}_{46,46}-\mathbf{I}_{52,52}+\mathbf{I}_{53,53} \\
& +\mathbf{I}_{54,54},
\end{aligned}
\tag{3.8}
$$

and the generators corresponding to simple roots are

$$
\begin{aligned}
\mathbf{e}_1 = {} & \mathbf{I}_{6,7}+\mathbf{I}_{30,8}+\mathbf{I}_{31,9}+\mathbf{I}_{19,20}+\mathbf{I}_{51,21}+\mathbf{I}_{52,22}+\mathbf{I}_{32,33}+\mathbf{I}_{55,34} \\
& +\mathbf{I}_{56,35}+\mathbf{I}_{41,42}+\mathbf{I}_{48,49}+\mathbf{I}_{53,54}, \\
\mathbf{e}_2 = {} & \mathbf{I}_{5,6}+\mathbf{I}_{9,10}+\mathbf{I}_{33,11}+\mathbf{I}_{18,19}+\mathbf{I}_{22,23}+\mathbf{I}_{54,24}+\mathbf{I}_{28,29}+\mathbf{I}_{34,43} \\
& +\mathbf{I}_{35,36}+\mathbf{I}_{40,41}+\mathbf{I}_{47,48}+\mathbf{I}_{50,51}, \\
\mathbf{e}_3 = {} & \mathbf{I}_{4,5}+\mathbf{I}_{8,9}+\mathbf{I}_{11,12}+\mathbf{I}_{43,13}+\mathbf{I}_{17,18}+\mathbf{I}_{21,22}+\mathbf{I}_{24,25}+\mathbf{I}_{30,31} \\
& +\mathbf{I}_{36,37}+\mathbf{I}_{39,40}+\mathbf{I}_{46,47}+\mathbf{I}_{51,52}, \\
\mathbf{e}_4 = {} & \mathbf{I}_{3,4}+\mathbf{I}_{9,33}+\mathbf{I}_{10,11}+\mathbf{I}_{13,14}+\mathbf{I}_{37,15}+\mathbf{I}_{18,50}+\mathbf{I}_{19,51}+\mathbf{I}_{20,21} \\
& +\mathbf{I}_{25,26}+\mathbf{I}_{31,32}+\mathbf{I}_{38,39}+\mathbf{I}_{45,46}, \\
\mathbf{e}_5 = {} & \mathbf{I}_{2,3}+\mathbf{I}_{11,43}+\mathbf{I}_{12,13}+\mathbf{I}_{15,16}+\mathbf{I}_{39,17}+\mathbf{I}_{40,18}+\mathbf{I}_{41,19}+\mathbf{I}_{42,20} \\
& +\mathbf{I}_{26,27}+\mathbf{I}_{32,55}+\mathbf{I}_{33,34}+\mathbf{I}_{44,45}, \\
\mathbf{e}_6 = {} & \mathbf{I}_{1,2}+\mathbf{I}_{13,37}+\mathbf{I}_{14,15}+\mathbf{I}_{27,28}+\mathbf{I}_{34,35}+\mathbf{I}_{43,36}+\mathbf{I}_{45,38}+\mathbf{I}_{46,39} \\
& +\mathbf{I}_{47,40}+\mathbf{I}_{48,41}+\mathbf{I}_{49,42}+\mathbf{I}_{54,55}, \\
\mathbf{e}_7 = {} & \mathbf{I}_{5,29}+\mathbf{I}_{1,30}+\mathbf{I}_{7,8}+\mathbf{I}_{12,44}+\mathbf{I}_{13,45}+\mathbf{I}_{14,46}+\mathbf{I}_{15,39}+\mathbf{I}_{16,17} \\
& +\mathbf{I}_{22,54}+\mathbf{I}_{23,24}+\mathbf{I}_{37,38}+\mathbf{I}_{52,53}.
\end{aligned}
\tag{3.9}
$$

3.2.5 An explicit form for E_8

The fundamental dominant weights of E_8 are $\lambda_1 = 4\alpha_1+7\alpha_2+10\alpha_3+8\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+5\alpha_8$, $\lambda_2 = 7\alpha_1+14\alpha_2+20\alpha_3+16\alpha_4+12\alpha_5+8\alpha_6+4\alpha_7+10\alpha_8$, $\lambda_3 = 10\alpha_1+20\alpha_2+30\alpha_3+24\alpha_4+18\alpha_5+12\alpha_6+6\alpha_7+15\alpha_8$, $\lambda_4 = 8\alpha_1+16\alpha_2+24\alpha_3+20\alpha_4+15\alpha_5+10\alpha_6+5\alpha_7+12\alpha_8$, $\lambda_5 = 6\alpha_1+12\alpha_2+18\alpha_3$

$+15\alpha_4+12\alpha_5+8\alpha_6+4\alpha_7+9\alpha_8$, $\lambda_6 = 4\alpha_1+8\alpha_2+12\alpha_3+10\alpha_4+8\alpha_5+6\alpha_6+3\alpha_7+6\alpha_8$, $\lambda_7 = 2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+3\alpha_8$, $\lambda_8 = 5\alpha_1+10\alpha_2+15\alpha_3+12\alpha_4+9\alpha_5+6\alpha_6+3\alpha_7+8\alpha_8$, and the dimensions of the fundamental representations are 3875, 6696000, 6899079264, 146325270, 2450240, 30380, 248, 147250. Thus the (0,0,0,0,0,0,1,0) representation is both the regular representation and the lowest dimensional representation. A matrix representation of (0,0,0,0,0,0,1,0) is given.

The Cartan subalgebra has as its basis the diagonal matrices

$$\underset{\sim}{h}_i = \sum_j a_{ij} \underset{\sim}{I}_{j,j} \qquad \text{for } i = 1,2,\ldots,8,\ j = 1,2,\ldots,248 \qquad (3.\ 10)$$

where the non-zero a_{ij} are given below:-

$$a_{1j} = \begin{cases} -2 & \text{for } j = 29, \\ +2 & \text{for } j = 31, \\ -1 & \text{for } j = 8,\ldots,13,24,\ldots,28,50,51,52,60,75,\ldots,78,84,\ldots,94, \\ & \quad 99,100,101,116,133,134,155,\ldots,160,172,178,\ldots,187, \\ & \quad 201,202,214, \\ +1 & \text{for } j = 14,32,\ldots,42,53,61,\ldots,66,79,\ldots,82,102,\ldots,106, \\ & \quad 117,\ldots,125,135,136,137,161,173,174,175,188,203,204, \\ & \quad 208,215,\ldots,221,235. \end{cases}$$

$$a_{2j} = \begin{cases} -2 & \text{for } j = 82, \\ +2 & \text{for } j = 84, \\ -1 & \text{for } j = 6,7,14,21,22,23,28,31,39,\ldots,42,48,49,53,63,\ldots,66,72, \\ & \quad 73,74,79,80,81,98,105,106,110,\ldots,115,121,\ldots,125,131,132, \\ & \quad 137,151,154,161,171,174,175,200,208,220,221,231,\ldots,234, \\ +1 & \text{for } j = 8,15,24,29,32,43,50,54,60,67,75,76,85,\ldots,90,99,100,101, \\ & \quad 107,108,126,127,133,134,138,139,140,145,146,147,155,156, \\ & \quad 157,162,163,164,176,\ldots,181,192,\ldots,195,201,202,209,214, \\ & \quad 222,236,237. \end{cases}$$

$$a_{3j} = \begin{cases} -2 & \text{for } j = 108, \\ +2 & \text{for } j = 110, \\ -1 & \text{for } j = 5,8,15,18,19,20,24,27,32,35,36,37,38,43,46,47,50,54, \\ & \quad 62,67,70,71,75,76,81,84,87,\ldots,90,103,104,107,126,127, \\ & \quad 130,136,140,146,147,150,157,162,163,164,167,\ldots,170, \\ & \quad 173,191,195,199,209,219,222, \\ +1 & \text{for } j = 6,9,16,21,25,28,33,39,44,48,51,55,63,68,72,73,77,82, \\ & \quad 85,91,98,105,111,121,122,123,128,131,132,137,141,148, \\ & \quad 151,154,158,165,172,174,175,182,183,184,189,196,200, \\ & \quad 205,210,212,220,223,224,225,231,\ldots,234. \end{cases}$$

$$a_{4j} = \begin{cases} -2 & \text{for } j = 165, \\ +2 & \text{for } j = 167, \\ -1 & \text{for } j = 4,9,16,17,21,25,26,33,34,39,44,45,48,55,63,68,69,72, \\ & \quad 73,80,85,86,91,100,101,102,105,110,111,122,123,128, \\ & \quad 129,134,135,139,145,148,149,156,164,177,183,184,188, \\ & \quad 193,194,198,205,218,223,224,225,228,229,230, \\ +1 & \text{for } j = 5,10,18,22,27,35,40,46,49,50,56,64,70,74,75,76,81,87, \\ & \quad 92,95,103,106,107,108,112,116,117,124,130,136,140,143, \\ & \quad 146,150,152,157,168,169,170,171,173,178,185,187,195, \\ & \quad 199,206,207,208,209,219,226,231,238,239,248. \end{cases}$$

$$a_{5j} = \begin{cases} -2 & \text{for } j = 226, \\ +2 & \text{for } j = 228, \\ -1 & \text{for } j = 3,10,18,22,35,40,46,56,64,70,76,77,78,79,87,92,95,98, \\ & \quad 99,103,112,117,121,124,132,133,140,141,142,143,146,152, \\ & \quad 154,155,163,167,175,176,178,182,185,192,195,196,197, \\ & \quad 204,212,213,216,217,225,231,238,239,242,243, \\ +1 & \text{for } j = 4,11,19,23,24,25,26,36,41,47,48,57,65,71,72,73,80,88, \\ & \quad 93,96,100,104,105,113,118,122,125,126,127,128,129,134, \\ & \quad 135,144,147,148,149,153,156,164,165,168,177,179,183, \\ & \quad 186,188,193,198,205,218,229,230,232,240,244. \end{cases}$$

$$a_{6j} = \begin{cases} -2 & \text{for } j = 240, \\ +2 & \text{for } j = 242, \\ -1 & \text{for } j = 2,11,19,36,41,57,65,73,74,75,88,93,96,100,113,118,122, \\ & \quad 127,\ldots,131,147,\ldots,151,160,161,162,168,172,173,174, \\ & \quad 179,183,187,188,193,202,203,207,\ldots,211,214,215,224, \\ & \quad 228,232,237,239,244,247,248, \\ +1 & \text{for } j = 3,12,20,21,22,37,42,\ldots,46,58,66,\ldots,70,76,\ldots,79,89, \\ & \quad 94,97,101,102,103,114,119,123,124,132,133,143,152,154, \\ & \quad 155,163,169,175,176,180,184,185,194,\ldots,197,204,216, \\ & \quad 225,226,229,233,243,245. \end{cases}$$

$$a_{7j} = \begin{cases} -2 & \text{for } j = 245, \\ +2 & \text{for } j = 247, \\ -1 & \text{for } j = 1,12,37,58,66,\ldots,72,89,114,119,123,\ldots,126,152,\ldots,159, \\ & \quad 169,171,180,184,185,186,194,\ldots,201,205,206,216, \\ & \quad 218,\ldots,223,229,233,235,236,238,242,244, \\ +1 & \text{for } j = 2,13,\ldots,19,38,\ldots,41,59,73,74,75,90,\ldots,93,95,96,98, \\ & \quad 99,100,115,120,127,\ldots,131,139,\ldots,142,160,161,162,170, \\ & \quad 172,173,174,187,188,202,203,207,217,224,230,234,237,239, \\ & \quad 240,243. \end{cases}$$

$$a_{8j} = \begin{cases} -2 & \text{for } j = 189, \\ +2 & \text{for } j = 191, \\ -1 & \text{for } j = 6,16,25,33,44,51,60,61,68,77,85,95,96,97,105,106,107, \\ & \quad 110,116,\ldots,120,128,137,138,141,143,144,148,152,153, \\ & \quad 158,172,174,\ldots,181,196,205,206,207,210,212,220,231, \\ & \quad 232,233,234,236,237,248, \\ +1 & \text{for } j = 7,8,17,\ldots,20,26,27,34,35,36,37,38,45,46,47,52,53,54, \\ & \quad 62,69,70,71,78,\ldots,81,86,\ldots,90,108,111,\ldots,115,129, \\ & \quad 130,142,149,150,159,\ldots,164,197,198,199,211,213,221, \\ & \quad 222. \end{cases}$$

3.3 Inner Involutive Automorphisms of Compact Exceptional Simple Lie Algebras and the real Forms generated by them

3.3.1 General theory

Any chief inner automorphism can be expressed as $T(\exp \underset{\sim}{h})$, where $\underset{\sim}{h}$ belongs to the Cartan subalgebra. Involutiveness implies that $T(\exp 2\underset{\sim}{h}) = T(\underset{\sim}{I})$. Since $T(\exp \underset{\sim}{h})$ can be expressed as

$$\underset{\sim}{I}\ (\ell) \oplus \sum_{\text{all } \alpha} \begin{pmatrix} \cosh\alpha(\underset{\sim}{h}), & -i \sinh\alpha(\underset{\sim}{h}) \\ +i\sinh\alpha(\underset{\sim}{h}), & \cosh\alpha(\underset{\sim}{h}) \end{pmatrix}$$

where ℓ is the rank of the algebra and α is a root, involutiveness implies that for every root α, $\alpha(\underset{\sim}{h}) = ni\pi$, where n is an integer. Thus a necessary and sufficient condition for involutiveness is that for every simple root α_j, $\alpha_j(\underset{\sim}{h}) = n_j i\pi$, n_j being an integer. If $\underset{\sim}{h} = (a_j i \underset{\sim}{h}_j)$, then this condition becomes $\alpha_i(a_j \underset{\sim}{h}_j) = n_i\pi$. This implies that $\sum_{j=1}^{\ell} a_j A_{ji} = n_i\pi$, $i = 1, \ldots, \ell$, where $\underset{\sim}{A}$ is the Cartan matrix, and hence $a_j = \pi \sum_{i=1}^{\ell} n_i (\underset{\sim}{A}^{-1})_{ij}$. (Gantmacher has shown [19,20] that some sets of n_i give equivalent automorphisms.) The inverses of the Cartan matrices are given in appendix 1. Thus the set of non-equivalent involutive inner automorphisms can be found. This method is now applied to all the exceptional simple Lie algebras.

3.3.2 G_2

Involutive chief inner automorphisms generating non-isomorphic real forms are obtained for (i) $n_1 = n_2 = 0$, and (ii) $n_1 = +2$, $n_2 = -1$, which give $\underset{\sim}{h} = \underset{\sim}{0}$ and $\underset{\sim}{h} = i\pi\underset{\sim}{h}_1$. The traces of $T(\exp i\underset{\sim}{h})$ are 14 and -2 respectively. Thus there are two non-isomorphic real forms of G_2.

(i) The compact form (generated by $T(\exp i\underset{\sim}{h})$ for $\underset{\sim}{h} = \underset{\sim}{0}$) is isomorphic to the set of orthogonal transformations T which leave the vector product in seven dimensions invariant (i.e. $T(\underset{\sim}{a} \times \underset{\sim}{b}) = T\underset{\sim}{a} \times T\underset{\sim}{b}$ for

$$(\underset{\sim}{a} \times \underset{\sim}{b})_i = \begin{vmatrix} a_{i-3}, & b_{i-3} \\ a_{i-2}, & b_{i-2} \end{vmatrix} + \begin{vmatrix} a_{i+2}, & b_{i+2} \\ a_{i-1}, & b_{i-1} \end{vmatrix} + \begin{vmatrix} a_{i+1}, & b_{i+1} \\ a_{i+3}, & a_{i+3} \end{vmatrix} \quad i = 1, \ldots, 7,$$

where all indices are reduced to mod 7. This group will henceforth be referred to as CG_2 and has a character of -14.

(ii) The non-compact form (generated by $T(\exp \underset{\sim}{h})$ for $\underset{\sim}{h} = i\pi\underset{\sim}{h}_1$) is isomorphic to the set of transformations which leave invariant the above vector product and the indefinite quadratic form $z^2 + x_1y_1 + x_2y_2 + x_3y_3$, i.e. $\underset{\sim}{X}\,\underset{\sim}{J}\,\tilde{\underset{\sim}{X}}$, where $\underset{\sim}{X} = (x_1, x_2, x_3, z, y_3, y_2, y_1)$ and $J_{ij} = \delta_{i,\,8-j}$ for $i, j = 1,\ldots,7$. This group will henceforth be referred to as NG_2 and has a character of 2.

3.3.3 F_4

Involutive chief inner automorphisms generating non-isomorphic real forms are obtained for (i) $n_1 = n_2 = n_3 = n_4 = 0$, (ii) $n_1 = +2$, $n_2 = -1$, $n_3 = n_4 = 0$, and (iii) $n_1 = 0$, $n_2 = -2$, $n_3 = +2$, $n_4 = -1$, which gives $\underset{\sim}{h} = 0$, $\underset{\sim}{h} = \pi i\underset{\sim}{h}_1$, and $\underset{\sim}{h} = \pi i\underset{\sim}{h}_3$ respectively. The traces of $T(\exp \underset{\sim}{h})$ are 52, -4, and 20 respectively.

These algebras are isomorphic to the groups of transformations in 26 variables given in §12 of [20] by Gantmacher and will be referred to as CF_4 (compact real form, generated by $T(\underset{\sim}{I})$, character -52), NF_4^1 (non-compact real form, generated by $T(\exp i\pi\underset{\sim}{h}_1)$, character +4) and NF_4^2 (non-compact real form, generated by $T(\exp i\pi\underset{\sim}{h}_3)$, character -20).

3.3.4 E_6

Involutive chief inner automorphisms generating non-isomorphic real forms are obtained for

	(i) $n_i = 0$	$i = 1,\ldots,6$	
	(ii) $n_i = 0$	$i \neq 4$,	$n_4 = -1$,
and	(iii) $n_i = 0$	$i \neq 5$,	$n_5 = -1$.

These give $\underset{\sim}{h} = \underset{\sim}{0}$, $\underset{\sim}{h} = \frac{\pi i}{3}(4\underset{\sim}{h}_1 + 8\underset{\sim}{h}_2 + 12\underset{\sim}{h}_3 + 10\underset{\sim}{h}_4 + 5\underset{\sim}{h}_5 + 6\underset{\sim}{h}_6)$ and $\frac{\pi i}{3}(2\underset{\sim}{h}_1 + 4\underset{\sim}{h}_2 + 6\underset{\sim}{h}_3 + 5\underset{\sim}{h}_4 + 4\underset{\sim}{h}_5 + 3\underset{\sim}{h}_6)$. The traces of $T(\exp \underset{\sim}{h})$ are then 78, -2 and 14 respectively. Thus there are three non-isomorphic real forms of E_6 generated by inner automorphisms.

(i) The compact form (generated by $T(\exp i\underset{\sim}{h})$ for $\underset{\sim}{h} = \underset{\sim}{0}$ is isomorphic to the set of transformations which leave invariant the positive definite hermitian form

$$x_p x_p^* + y_q y_q^* + z_{pq}\, z_{pq}^* = \underset{\sim}{X}.\tilde{\underset{\sim}{X}} \tag{3.11}$$

for $p,q = 1,\ldots,6$, $z_{pq} = -z_{qp}$ and

$\underset{\sim}{X} = (x_1, x_2, x_3, x_4, x_5, x_6, 2z_{5,6}, 2z_{4,6}, 2z_{3,6}, 2z_{2,6}, 2z_{4,5}, 2z_{3,5}, 2z_{3,4}, 2z_{2,5}, 2z_{1,6}, 2z_{2,4}, 2z_{2,3}, 2z_{1,5}, 2z_{1,4}, 2z_{1,3}, 2z_{1,2}, y_1, y_2, y_3, y_4, y_5, y_6)$ and the cubic form in 27 variables

$$x_p y_q z_{pq} - \epsilon(pqstuv)\ z_{pq} z_{st} z_{uv} \equiv g_{ijk} X_i X_j X_k , \qquad (3.12)$$

where p, q, s, t, u, v, i, j, k = 1, ..., 6, X_i is the ith element of $\underset{\sim}{X}$, and ϵ(pqstuv) is + 1 for an even permutation of (1, 2, 3, 4, 5, 6), -1 for an odd permulation and 0 otherwise. The non-zero elements of g_{ijk} are listed below.

g_{ijk} = +1 for {i,j,k} = {1, 23, 21}, {1, 24, 20}, {1, 25, 19}, {1, 26, 18}, {1, 27, 15}, {2, 24, 17}, {2, 25, 16}, {2, 26, 14}, {2, 27, 10}, {3, 25, 13} {3, 26, 12} {3, 27, 9 }, {4, 26, 11}, {4, 27, 8}, {5, 27, 7}, {7, 13, 21}, {7, 17, 19}, {8, 14, 20}, {9, 11, 21}, {9, 16, 18}, {10, 12, 19}, {11, 15, 17}, {13, 14, 15}

g_{ijk} = -1 for {i,j,k} = {2, 22, 21}, {3, 22, 20}, {3, 23, 17}, {4, 22, 19}, {4, 23, 16}, {4, 24, 13}, {5, 22, 18}, {5, 23, 14}, {5, 24, 12}, {5, 25, 11}, {6, 22, 15}, {6, 23, 10}, {6, 24, 9}, {6, 25, 8}, {6, 26, 7}, {7, 16, 20}, {8, 12, 21}, {8, 17, 18}, {9, 14, 19}, {10, 11, 20}, {10, 13, 18}, {12, 15, 16}

This algebra has a character of -78 and will be referred to as CE_6.

(ii) The non-compact real form generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \frac{\pi i}{3}(4\underset{\sim}{h}_1 + 8\underset{\sim}{h}_2 + 12\underset{\sim}{h}_3 + 10\underset{\sim}{h}_4 + 5\underset{\sim}{h}_5 + 6\underset{\sim}{h}_6)$ is isomorphic to the group of transformations in 27 variables which leave invariant the ~~positive~~ definite hermitian form

$$x_p x_p^* + y_q y_q^* - z_{pq} z_{pq}^* = \underset{\sim}{X}\, \underset{\sim}{J}'\, \tilde{\underset{\sim}{X}}^* \qquad (3.13)$$

for p,q = 1, ..., 6 and $J'_{ij} = \begin{cases} \delta_{ij} & \text{for } i \le 6 \text{ and } i \ge 22, \\ -\delta_{ij} & \text{for } 6 < i < 22, \end{cases}$

and the cubic form (3.12). This algebra will be referred to as NE_6^1. It has character 2.

(iii) The non-compact real form generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \frac{\pi i}{3}(2\underset{\sim}{h}_1 + 4\underset{\sim}{h}_2 + 6\underset{\sim}{h}_3 + 5\underset{\sim}{h}_4 + 4\underset{\sim}{h}_5 + 3\underset{\sim}{h}_6)$ is isomorphic to the group of transformations in 27 variables which leaves invariant the ~~positive~~ definite hermitian form

$$x_p x_p^* - x_6 x_6^* - y_q y_q^* + y_6 y_6^* + z_{pq} z_{pq}^* - z_{p6} z_{p6}^* - z_{6q} z_{6q}^* = \underset{\sim}{X}\, \underset{\sim}{J}''\, \tilde{\underset{\sim}{X}}^* \qquad (3.14)$$

for p,q = 1,...,5 and $(J'')_{ij} = \begin{cases} \delta_{ij} & i = 1,\ldots,5,11,\ldots,14,16,\ldots,21,27, \\ -\delta_{ij} & i = 6,7,\ldots,10,15,22,\ldots,26, \end{cases}$

and the cubic form (3.12). This algebra will be referred to as NE_6^2 and has character-14.

3.3.5 E_7

Non-isomorphic real forms of E_7 are generated from involutive chief inner automorphisms corresponding to

i) $n_i = 0$, $i = 1,\ldots,7$,

ii) $n_i = 0$, $i \neq 6$, $n_6 = -1$,

iii) $n_i = 0$, $i \neq 7$, $n_7 = -1$,

iv) $n_i = 0$, $i \neq 6,7, n_6 = -1$, $n_7 = +1$.

This gives i) $\underset{\sim}{h} = \underset{\sim}{0}$, ii) $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+3\underset{\sim}{h}_3+\frac{5}{2}\underset{\sim}{h}_4+2\underset{\sim}{h}_5+\frac{3}{2}\underset{\sim}{h}_6+\frac{3}{2}\underset{\sim}{h}_7)$, iii) $\underset{\sim}{h} = \pi i (2\underset{\sim}{h}_1+4\underset{\sim}{h}_2+6\underset{\sim}{h}_3+\frac{9}{2}\underset{\sim}{h}_4+3\underset{\sim}{h}_5+\frac{3}{2}\underset{\sim}{h}_6+\frac{7}{2}\underset{\sim}{h}_7)$, iv) $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+3\underset{\sim}{h}_3+2\underset{\sim}{h}_4+\underset{\sim}{h}_5+2\underset{\sim}{h}_7)$. The traces of $T(\exp \underset{\sim}{h})$ are 133, -5, 7, -25 respectively. Thus there are four non-isomorphic real forms of E_7 generated by inner automorphisms:

i) The compact form (generated by $T(\exp \underset{\sim}{h})$ for $\underset{\sim}{h} = \underset{\sim}{0}$) is isomorphic to the group of transformations in 56 variables $x_{pq} = -x_{qp}$ and $y_{pq} = -y_{qp}$, $p,q = 1,\ldots,8$, which leave invariant the positive definite Hermitian form

$$x_{pq}\, x^*_{pq} + y_{pq}\, y^*_{pq}\ , \tag{3.15}$$

the bilinear form

$$x_{pq}\, y^*_{pq} - y_{pq}\, x^*_{pq} \tag{3.16}$$

for $p,q = 1,\ldots,8$, and the biquadratic form

$$x_{pq}\, x_{rs}\, y_{ps}\, y_{qr} + \epsilon(pqrstuvw)(x_{pq}\, x_{rs}\, x_{tu}\, x_{vw} + y_{pq}\, y_{rs}\, y_{tu}\, y_{vw}). \tag{3.17}$$

This algebra will be referred to as CE_7, and has character -133.

ii) The non-compact real form generated by $T(\exp \underset{\sim}{h})$ for $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+3\underset{\sim}{h}_3+\frac{5}{2}\underset{\sim}{h}_4+2\underset{\sim}{h}_5+\frac{3}{2}\underset{\sim}{h}_6+\frac{3}{2}\underset{\sim}{h}_7)$ is isomorphic to the group of transformations in 56 variables which leave invariant the forms (3.16) and (3.17) and the indefinite Hermitian form

$$\sum_{pq} \lambda_p\, \lambda_q (x_{pq}\, x^*_{pq} + y_{pq}\, y^*_{pq}) \tag{3.18}$$

for $\lambda_1 = \lambda_2 = -1$, $\lambda_p = 1$, $p \neq 1,2$.

This algebra will be referred to as NE^1_7 and has character $+5$.

iii) The non-compact real form generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \pi i(2\underset{\sim}{h}_1+4\underset{\sim}{h}_2+6\underset{\sim}{h}_3+\frac{9}{2}\underset{\sim}{h}_4+3\underset{\sim}{h}_5+\frac{3}{2}\underset{\sim}{h}_6+\frac{7}{2}\underset{\sim}{h}_7)$ is isomorphic to the group of transformations in 56 real variables leaving invariant the forms(3.16) and(3.17). This algebra will be referred to as NE_7^2 and has a character of -7.

iv) The non-compact real form generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+3\underset{\sim}{h}_3+2\underset{\sim}{h}_4+\underset{\sim}{h}_5+2\underset{\sim}{h}_7)$ is isomorphic to the group of transformations in 56 variables x_{pq} and y_{pq} connected by

$$y_{pq} = \lambda_p \lambda_q x^*_{pq} \qquad (\lambda_1 = \lambda_2 = -1, \lambda_p = 1, p \neq 1,2)$$

which leave invariant the forms(3.16) and(3.17).

This algebra will be referred to as NE_7^3 and has character +25.

3.3.6 E_8

Non-isomorphic real forms of E_8 are generated by involutive inner automorphisms corresponding to

i) $n_i = 0$, for all i,

ii) $n_i = 0$, $i \neq 5$, $n_5 = -1$,

iii) $n_i = 0$, $i \neq 6$, $n_6 = -1$,

This gives i) $\underset{\sim}{h} = \underset{\sim}{0}$, ii) $\underset{\sim}{h} = \pi i(6\underset{\sim}{h}_1+12\underset{\sim}{h}_2+18\underset{\sim}{h}_3+15\underset{\sim}{h}_4+12\underset{\sim}{h}_5+8\underset{\sim}{h}_6+4\underset{\sim}{h}_7+9\underset{\sim}{h}_8)$, and iii) $\underset{\sim}{h} = \pi i(4\underset{\sim}{h}_1+8\underset{\sim}{h}_2+12\underset{\sim}{h}_3+10\underset{\sim}{h}_4+8\underset{\sim}{h}_5+6\underset{\sim}{h}_6+3\underset{\sim}{h}_7+6\underset{\sim}{h}_8)$. This implies that there are three real forms of E_8 generated by inner automorphisms.

(i) The compact real form generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \underset{\sim}{0}$, has character -248 and will be referred to as CE_8.

(ii) The non-compact real form generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \pi i(6\underset{\sim}{h}_1+12\underset{\sim}{h}_2+18\underset{\sim}{h}_3+15\underset{\sim}{h}_4+12\underset{\sim}{h}_5+8\underset{\sim}{h}_6+4\underset{\sim}{h}_7+8\underset{\sim}{h}_8)$ has character +8 and will be referred to as NE_8^1.

(iii) The other non-compact real form, generated by T(exp $\underset{\sim}{h}$) for $\underset{\sim}{h} = \pi i(4\underset{\sim}{h}_1+8\underset{\sim}{h}_2+12\underset{\sim}{h}_3+10\underset{\sim}{h}_4+8\underset{\sim}{h}_5+6\underset{\sim}{h}_6+3\underset{\sim}{h}_7+6\underset{\sim}{h}_8)$ has character -24 and will be referred to as NE_8^2.

3.4. Outer Involutive Automorphism of E_6 and the real forms generated by them

3.4.1 Chief outer autmophisms of E_6

As stated in [11] every outer involutive automorphism of $\mathcal{L}_c$ can be expressed as $A = U Z_o \exp(\text{ad } \underset{\sim}{h}) U^{-1}$, where U is an inner automorphism of $\mathcal{L}_c$ and $Z_o \exp(\text{ad } \underset{\sim}{h})$ is a chief outer automorphism ($Z_o \underset{\sim}{h} = \underset{\sim}{h}$). There is one Z_o (not uniquely defined) corresponding to each connected component of the group of automorphisms. Grantmacher [20] has shown that E_6 has two such connected components, one containing the inner automorphism ($Z_o = I$) and one containing the outer automorphisms. Z_o corresponds to a 'particular rotation' $\underset{\sim}{\tau}$ (as defined by Grantmacher [19]) of the root space. In the case of E_6, $\underset{\sim}{\tau}$ is such that

$$\left.\begin{aligned} \underset{\sim}{\tau}\alpha_i &= \alpha_{6-i}\,, \qquad i = 1, .., 5, \\ \underset{\sim}{\tau}\alpha_6 &= \alpha_6\,. \end{aligned}\right\} \tag{3.19}$$

Hence Z_o must satisfy

$$\left.\begin{aligned} Z_o\underset{\sim}{h}_i &= \underset{\sim}{h}_{6-i}\,, \qquad i = 1, ..5\,, \\ Z_o\underset{\sim}{h}_6 &= \underset{\sim}{h}_6\,, \\ Z_o\underset{\sim}{e}_i &= \underset{\sim}{e}_{6-i}\,, \qquad i = i, ...5\,, \\ Z_o\underset{\sim}{e}_6 &= \underset{\sim}{e}_6\,. \end{aligned}\right\} \tag{3.20}$$

This rotation τ is associated with the transformation $\alpha_j \to \alpha_{6-j}$, $j = 1, \ldots, 5$, $\alpha_6 \to \alpha_6$, that leaves invariant the Dynkin diagram for E_6. The outer automorphism of D_ℓ can be expressed in the form $T(\underset{\sim}{q})$ (where $\underset{\sim}{q}$ does not belong to $SO(2\ell)$), but here there is no $\underset{\sim}{q}$ for which $Z_o = T(\underset{\sim}{q})$ is consistent with (3.20), so Z_o must be expressed in a different form.

3.4.2 An explicit form for Z_o in terms of K

Since in [30] Mehta and Srivastava show that the twenty seven dimensional representation of E_6 is not equivalent to its complex conjugate, the operation of complex conjugation K must be an outer automorphism. It must therefore be possible to express Z_o as

$$Z_o = T(\underset{\sim}{d})K, \tag{3.21}$$

as was done in the case of A_ℓ in [11], where $\underset{\sim}{d} \in G_c = CE_6$. K is defined by

$$\left.\begin{aligned} K(i\,\underset{\sim}{h}_j) &= -i\,\underset{\sim}{h}_j\,, \\ K(\underset{\sim}{e}_\alpha + \underset{\sim}{f}_\alpha) &= \underset{\sim}{e}_\alpha + \underset{\sim}{f}_\alpha\,, \\ Ki(\underset{\sim}{e}_\alpha - \underset{\sim}{f}_\alpha) &= -i(\underset{\sim}{e}_\alpha - \underset{\sim}{f}_\alpha). \end{aligned}\right\} \qquad (3.22)$$

Hence $\underset{\sim}{d}$ satisfies

$$\left.\begin{aligned} \underset{\sim}{d}\,\underset{\sim}{h}_i &= -\underset{\sim}{h}_{6-i}\,\underset{\sim}{d}\,, \quad i = 1,\dots,5, \\ \underset{\sim}{d}\,\underset{\sim}{h}_6 &= -\underset{\sim}{h}_6\,\underset{\sim}{d}, \\ \underset{\sim}{d}\,\underset{\sim}{e}_i &= \underset{\sim}{f}_{6-i}\,\underset{\sim}{d}\,, \quad i = 1,\dots,5, \\ \underset{\sim}{d}\,\underset{\sim}{f}_i &= \underset{\sim}{e}_{6-i}\,\underset{\sim}{d}\,, \quad i = 1,\dots,5, \\ \underset{\sim}{d}\,\underset{\sim}{e}_6 &= \underset{\sim}{f}_6\,\underset{\sim}{d}\,, \\ \underset{\sim}{d}\,\underset{\sim}{f}_6 &= \underset{\sim}{e}_6\,\underset{\sim}{d}\,. \end{aligned}\right\} \qquad (3.23)$$

One solution of (3.23) is the symmetric matrix

$$\underset{\sim}{d} = \Big(\sum_{j=13}^{15} -\underset{\sim}{I}_{jj} + \sum_{j=1}^{12} (-1)^j (+\underset{\sim}{I}_{j,28-j} + \underset{\sim}{I}_{28-j,j}) \Big). \qquad (3.24)$$

3.4.3 Real forms generated by outer involutive automorphisms

Since $A = U\,Z_o \exp(\mathrm{ad}\,\underset{\sim}{h})\,U^{-1}$ generates the same real form for all U belonging to the group of inner automorphisms of $\mathcal{L}_c$, it is only necessary to consider the case of $U = I$. Since $Z_o\underset{\sim}{h} = \underset{\sim}{h}$, $\underset{\sim}{h}$ must be of the form

$$\underset{\sim}{h} = \lambda_1(\underset{\sim}{h}_1 + \underset{\sim}{h}_5) + \lambda_2(\underset{\sim}{h}_2 + \underset{\sim}{h}_4) + \lambda_3\,\underset{\sim}{h}_3 + \lambda_6\,\underset{\sim}{h}_6.$$

We require that $Z_o \exp(\mathrm{ad}\,\underset{\sim}{h})$ must be involutive and since Z_o is involutive, the condition $Z_o\underset{\sim}{h} = \underset{\sim}{h}$ implies $\exp(\mathrm{ad}\,\underset{\sim}{h})$ must be involutive also. Thus $\exp(\mathrm{ad}\,\underset{\sim}{h})$ is an involutive chief inner automorphism. By reasoning similar to that in section 3.3.1, one obtains

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_2, \lambda_1, \lambda_6) = \pi i (n_1, n_2, n_3, n_4, n_5, n_6)\,\underset{\sim}{A}^{-1} \qquad (3.25)$$

where $n_i\,(i = 1,\dots,6)$ are integers and $\underset{\sim}{A}^{-1}$ is the inverse of the Cartan matrix (see A15). Gantmacher [20] has shown that only two sets of n_i in (3.25) lead to the generation of non-isomorphic real forms, namely when i) $n_i = 0$, $i = 1,\dots,6$ and ii) $n_i = 0$, $i = 1,\dots,5$, $n_6 = -1$ which give i) $\underset{\sim}{h} = \underset{\sim}{0}$ and ii) $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 + 3\underset{\sim}{h}_3 + 2\underset{\sim}{h}_4 + \underset{\sim}{h}_5 + 2\underset{\sim}{h}_6)$.

The automorphism $Z_o \exp(\text{ad } \underset{\sim}{h})$ for $\underset{\sim}{h} = \underset{\sim}{0}$ maps 26 generators into themselves namely $\underset{\sim}{h}_3$, $\underset{\sim}{h}_6$ and the generators $\underset{\sim}{e}_\alpha$ and $\underset{\sim}{f}_\alpha$ associated with the following roots α_3, α_6, $\alpha_3+\alpha_6$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_2+\alpha_3+\alpha_4+\alpha_6$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\alpha_2+2\alpha_3+\alpha_4+\alpha_6$, $\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$. Thus Z_o has a trace of +26 and the real form generated by it has a character of −26. This real form, henceforth referred to as NE_6^3 is isomorphic to the group of transformations in 27 complex variables x_p, y_q, $z_{pq}(= -z_{qp})$ for $p,q = 1,\ldots,6$, which leave invariant the cubic form

$$x_p \, y_q \, z_{pq} - \epsilon(pqstuv) \, z_{pq} \, z_{st} \, z_{uv} \; , \qquad (3.26)$$

the variables being subject to the conditions

$$y_{2p-1} = x^*_{2p} \; , \quad y_{2p} = - x^*_{2p-1} \; , \quad z_{2p-1,\, 2q-1} = z^*_{2p,2q} \; ,$$

$$z_{2p-1,\, 2q} = z^*_{2q-1,\, 2p} \; .$$

The automorphism $Z_o \exp(\text{ad } \underset{\sim}{h})$ for $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+3\underset{\sim}{h}_3+2\underset{\sim}{h}_4+\underset{\sim}{h}_5+2\underset{\sim}{h}_6)$ maps $\underset{\sim}{h}_3$ and $\underset{\sim}{h}_6$ and the generators $\underset{\sim}{e}_\alpha$ and $\underset{\sim}{f}_\alpha$ associated with α_3, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$ into themselves, and multiplies the generators associated with the following roots by (−1): α_6, $\alpha_3+\alpha_6$, $\alpha_2+\alpha_3+\alpha_4+\alpha_6$, $\alpha_2+2\alpha_3+\alpha_4+\alpha_6$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha_6$. Thus the automorphism has trace −6. Hence the real form generated by $Z_o \exp(\text{ad } \underset{\sim}{h})$ for $\underset{\sim}{h} = \pi i(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+3\underset{\sim}{h}_3+2\underset{\sim}{h}_4+\underset{\sim}{h}_5+2\underset{\sim}{h}_6)$ has character +6 and will be referred to as NE_6^4. It is isomorphic to the group of transformations in 27 real variables x_p, y_q, $z_{pq}(= -z_{qp})$ which leave invariant the form (3.26).

3.4.4 Extensions of Z_o

As $Z_o = T(\underset{\sim}{d})K$ for E_6, which is exactly the same as for A_ℓ, the theory of extensions of Z_o for E_6 goes through in the same way as that described in the latter part of section 2 of [11]. The analogues of equations (9), (10) and (16)

$$\underset{\sim}{d}^2 = \underset{\sim}{I}(27), \tag{3.26A}$$

(which follows from (3.24)),

$$\underset{\sim}{d}(\exp \underset{\sim}{h}) = \begin{cases} \exp(\underset{\sim}{h})\underset{\sim}{d} & \text{for } \underset{\sim}{h} = 0 \text{ (i.e. for } \mathcal{L} = NE_6^3) \\ -\exp(\underset{\sim}{h})\underset{\sim}{d} & \text{for } \underset{\sim}{h} \neq 0 \text{ (i.e. for } \mathcal{L} = NE_6^4) \end{cases} \tag{3.27}$$

(which follows from (3.23)), and

$$\underset{\sim}{\Gamma}(\underset{\sim}{d}')(\exp \underset{\sim}{\Gamma}(\underset{\sim}{h}'))*\underset{\sim}{\Gamma}(\underset{\sim}{d}')*(\exp(\underset{\sim}{\Gamma}(\underset{\sim}{h}')) = \begin{cases} \underset{\sim}{\Gamma}(\underset{\sim}{I}(27)), & \mathcal{L}' = NE_6^3 \\ \underset{\sim}{\Gamma}(-\underset{\sim}{I}(27)), & \mathcal{L}' = NE_6^4 \end{cases} . \tag{3.28}$$

3.5. Embeddings of a real Lie algebra $\mathcal{L}'$ in a real Lie algebra $\mathcal{L}$, when $\mathcal{L}'$ and/or $\mathcal{L}$ may be exceptional

3.5.1 General Theory for exceptional simple Lie algebras

The methods for embedding real simple Lie algebras in real simple Lie algebras, given in [10, 11, 12], require modification if $\mathcal{L}'$ or $\mathcal{L}$ is exceptional, although the basic principles are the same.

The necessary and sufficient condition (1.2) holds for any semi-simple real Lie algebra. Assuming $\underset{\sim}{\Gamma}$ can be reduced to the form

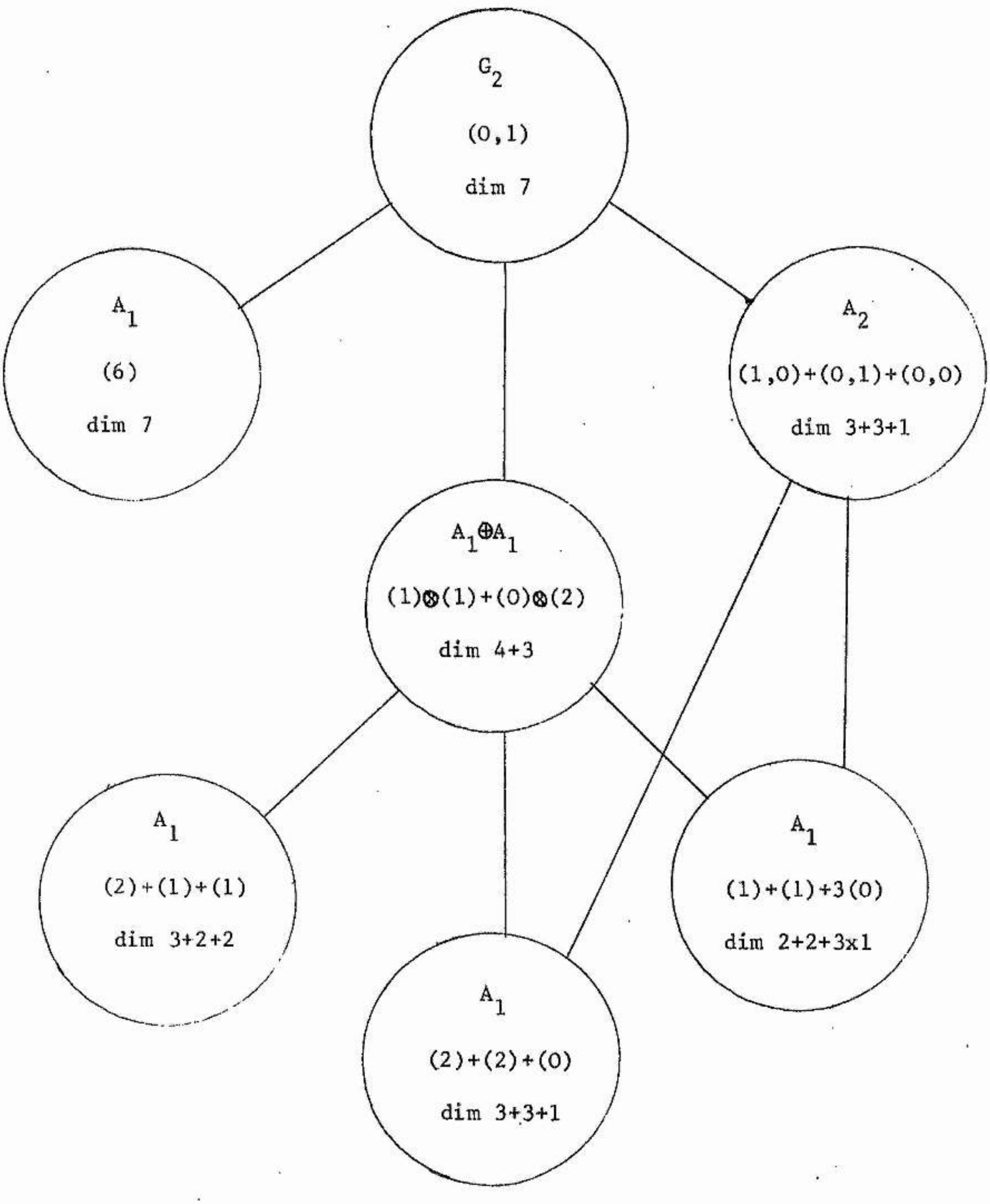
$$\underset{\sim}{\Gamma} = \underset{\sim}{B}\left(\oplus \sum_{j=1}^{n} (\underset{\sim}{I}(p_j) \otimes \underset{\sim}{\Gamma}^{j})\right)\underset{\sim}{B}^{-1}, \tag{3.29}$$

(the $\underset{\sim}{\Gamma}^{j}$ being irreducible representations of $\mathcal{L}_c$, occurring p_j times in the reduction of $\underset{\sim}{\Gamma}$) the determination of $\underset{\sim}{B}$ may not be straightforward, and $\underset{\sim}{B}$ may not necessarily belong to $\mathcal{L}_c$. If $\mathcal{L}$ is classical then the determination of B is easier. It may either be taken as $\underset{\sim}{I}$ without loss of generality, or is given in [10 or 12]. Moreover, if $\mathcal{L}$ is classical, the centralizer may also be calculated as in [9], as the nature of the subalgebra does not enter into the argument. A necessary condition for $\mathcal{L}'$ to be a subalgebra of $\mathcal{L}$ is that $\mathcal{L}_c{}'$ be a subalgebra of $\mathcal{L}_c$, but this is by no means sufficient. It is possible using the results of Patera and Sankoff [33] on maximal subalgebras of compact Lie algebras to list all semi-simple subalgebras of a given compact Lie algebra, together with the reduction of all possible embedding representations by constructing a "tree". For example all the semi-simple subalgebras of G_2 may be found from the tree in figure 3.1. Each branching process gives all maximal semi-simple subalgebras of previous algebras and the reduction of the representations. The branching process continues until there are no non-trivial maximal subalgebras of the last subalgebras (i.e. A_1); then all subalgebras have been included.

Unfortunately it is not easy to deduce the maximal subalgebras of non-compact real Lie algebras from these results. A maximal subalgebra $\mathcal{L}_c^{max}$ of the compact real form $\mathcal{L}_c$ of a complex Lie algebra $\tilde{\mathcal{L}}$ may not have the same complex extension $\tilde{\mathcal{L}}^{max}$ as any of the maximal subalgebras of a non-compact real form $\mathcal{L}$ of $\tilde{\mathcal{L}}$. For example take $\tilde{\mathcal{L}} = D_6$ (1,0,0,0,0,0), ($\mathcal{L}_c$ = SO(12)), $\mathcal{L}_c^{max}$ = SO(11) ((1,0,0,0,0) + (0,0,0,0,0)), ($\tilde{\mathcal{L}}^{max} = B_5$) and $\mathcal{L} = ND_{12}$. Then there is no real

Figure 3.1 Subalgebras of G_2 (0,1)

G_2 has 6 non-equivalent complex Lie subalgebras. (CG_2 has 6 non-equivalent compact real Lie subalgebras). The reduction of the representation of each subalgebra and its dimensions are given.



form of B_5 ((1,0,0,0,0) + (0,0,0,0,0)) which is a subalgebra of ND_{12} (see section 7.4 in [10] . ND_{10} ((1,0,0,0,0) + (0,0,0,0,0) + (0,0,0,0,0)) is however a maximal subalgebra of ND_{12}, even though SO(10) is not a maximal subalgebra of SO(12).

In order to find out if $\mathcal{L}_c'$ is a subalgebra of $\mathcal{L}_c$ the reverse of the branching process may be used, as in figure 3.2 . Once it has been established that $\mathcal{L}_c'$ can be embedded in $\mathcal{L}_c$ with a representation $\underset{\sim}{\Gamma}$ which reduces as in (3.29), $\underset{\sim}{B}$ being unknown, there are three main calculations:- the explicit form of $\underset{\sim}{\Gamma}$ (usually involving $\underset{\sim}{B}$), the extension of S' (the automorphism which generates $\mathcal{L}'$) and the calculation of the centralizer. Then it is straightforward to substitute into (1.2) in a similar way to that in [10, 11 and 12].

3.5.2 General Theory for exceptional semi-simple Lie algebras

The results of Ch2 are mostly valid for exceptional Lie algebras. The cases involving E_6, follow through in the way described for A_ℓ, with SL(n,R) replaced by NE_6^3 and $Q_{\frac{1}{2}(n+1)}$ by NE_6^4 (and the dimensions adjusted appropriately).

3.5.3 An explicit form for $\underset{\sim}{\Gamma}$

In cases where there is only one non-conjugate embedding of $\mathcal{L}_c'$ in $\mathcal{L}_c$, it may be possible to find an explicit form for $\underset{\sim}{\Gamma}$ unambiguously from the Dynkin diagram (e.g. for an embedding of E_6 in E_7 $\underset{\sim}{\Gamma}(\underset{\sim}{h}_i') = \underset{\sim}{h}_i$ $i = 1,\ldots,5$ and $\underset{\sim}{\Gamma}(\underset{\sim}{h}_6') = \underset{\sim}{h}_7$). Otherwise a solution of

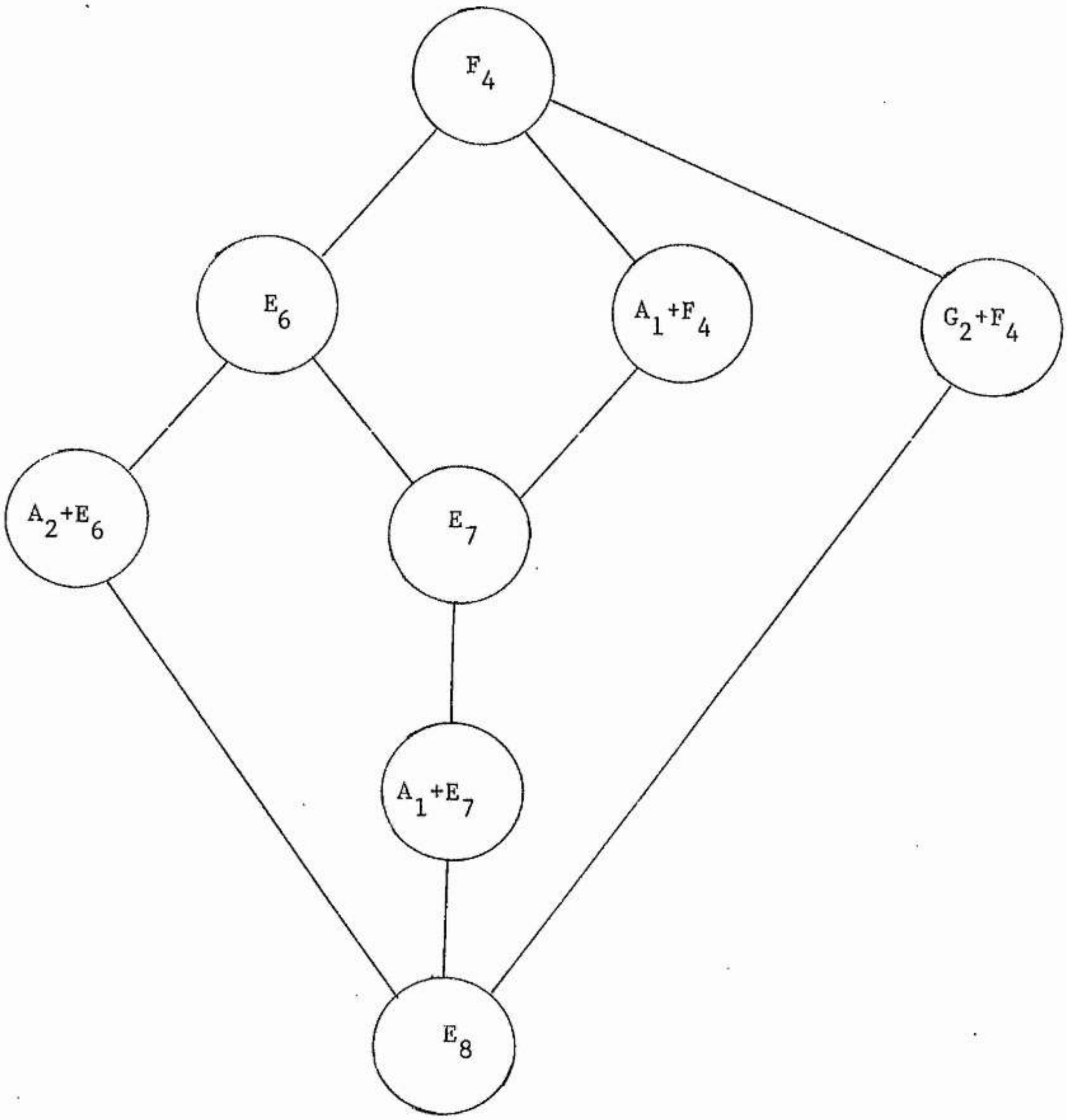
$$\underset{\sim}{B}\left(\oplus \sum_j (\underset{\sim}{I}(p_j) \otimes \underset{\sim}{\Gamma}^j(\underset{\sim}{b}_i'))\right)\underset{\sim}{B}^{-1} = a_{ik}\, \underset{\sim}{b}_k \tag{3.30}$$

is required (here $\underset{\sim}{b}_i'$ and $\underset{\sim}{b}_k$ are the generators of $\mathcal{L}_c'$ and $\mathcal{L}_c$). In some cases a solution for the a_{ik} may be found by inspection, but the equations (3.30) are under-determined. (This is to be expected as a result of the existence of conjugate embeddings $\underset{\sim}{B}'(\oplus \sum_j (\underset{\sim}{I}(p_j) \otimes \underset{\sim}{\Gamma}^j)\underset{\sim}{B}'^{-1}$, where $T(\underset{\sim}{B}\,\underset{\sim}{B}'^{-1})$ is an inner automorphism of $\mathcal{L}_c$. There are however a few subalgebras of D_ℓ and E_6 (see [13] and [14]) which have the same reduction and are not equivalent (i.e. $T(\underset{\sim}{B}\,\underset{\sim}{B}'^{-1})$ is not an inner automorphism). In such cases (3.30) must be solved with additional restraints.

Figure 3.2

Complex Lie algebras (rank $\leqslant$ 8) containing F_4

The compact real forms of these Lie algebras contain CF_4. If $\tilde{\mathcal{L}}'$ is any of the above Lie algebras and $\tilde{\mathcal{L}}''$ any other Lie algebra, then $\tilde{\mathcal{L}}' \oplus \tilde{\mathcal{L}}''$ contains F_4.



It is often simpler to solve (3.30) by considering the subset of the equations for which the $\underset{\sim}{b}_i'$ are generators of the Cartan subalgebra. Since $T(\exp i\underset{\sim}{h}')$ is an inner automorphism, $T(\exp \underset{\sim}{\Gamma}(i\underset{\sim}{h}'))$ must also be an inner automorphism. Hence with a suitable choice of $\underset{\sim}{B}$, $\underset{\sim}{\Gamma}(i\underset{\sim}{h}')$ can be chosen to belong to the Cartan subalgebra of $\mathcal{L}_c$ without loss of generality. The equations

$$\underset{\sim}{B}(\oplus \sum_j \underset{\sim}{I}(p_j) \otimes \underset{\sim}{\Gamma}^j(\underset{\sim}{h}_i')) = a_{ik} \underset{\sim}{h}_k \underset{\sim}{B} \tag{3.31}$$

can often be solved by inspection, by considering the symmetry properties of the Cartan subalgebra (e.g. certain elements always zero, antisymmetry about the alternate diagonal) and the interchange of rows and columns. Unless there is more than one non-conjugate embedding with the same reduction, a solution thus obtained may be used without loss of generality and any remaining arbitrary constants chosen as convenient.

3.5.4 The Centralizer $C(G_c', G_c)$

With $\underset{\sim}{\Gamma}$ expressed in the form (3.29) the centralizer elements $\underset{\sim}{c}$ will be of the form

$$\underset{\sim}{c} = \underset{\sim}{B}(\oplus \sum_j (\underset{\sim}{c}_j''(p_j) \otimes \underset{\sim}{I}(n_j)))\underset{\sim}{B}^{-1}, \tag{3.32}$$

where $\underset{\sim}{c}_j''$ is a p_j by p_j matrix. By assumption, $\underset{\sim}{c}$ belongs to G_c and can thus be written $\underset{\sim}{c} = \exp \lambda_i \underset{\sim}{b}_i$ ($\underset{\sim}{b}_i$ being the generators of $\mathcal{L}_c$), which places some restrictions on the $\underset{\sim}{c}_j''$. These restrictions may often be found using the following relationships.

(i) $\exp(\underset{\sim}{a} \oplus \underset{\sim}{b} \oplus \underset{\sim}{c} \ldots) = \exp \underset{\sim}{a} \oplus \exp \underset{\sim}{b} \oplus \exp \underset{\sim}{c} \ldots$ (3.33)

(ii) $\exp(\underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1}) = \underset{\sim}{B}(\exp \underset{\sim}{A})\underset{\sim}{B}^{-1}$. (3.34)

(iii) If $\underset{\sim}{A}^2 = -\underset{\sim}{I}$ (i.e. $\underset{\sim}{A} = -\underset{\sim}{A}^{-1}$), then

$$\exp \lambda\underset{\sim}{A} = \underset{\sim}{I} \cos \lambda + \underset{\sim}{A} \sin \lambda. \tag{3.35}$$

(iv) If $\underset{\sim}{A}^3 = -\underset{\sim}{A}$ and $\det \underset{\sim}{A} = 0$ (if $\underset{\sim}{A}^{-1}$ exists then (35) holds), then

$$\exp \lambda\underset{\sim}{A} = \underset{\sim}{I} + \underset{\sim}{A} \sin \lambda + \underset{\sim}{A}^2(1-\cos \lambda). \tag{3.36}$$

As shown in [9], $\underset{\sim}{c}$ can often be diagonalised into the form

$$\mathbf{c} = \mathbf{r}(\exp \mathbf{g})\mathbf{r}^{-1} , \qquad (3.37)$$

where $\mathbf{r}$ is a member of the centralizer and $\mathbf{g}$ a member of the Cartan subalgebra. This is always possible when $\mathbf{c}$ can be expressed as $\mathbf{B}(\oplus \sum_i \mathbf{c}_i(n_i))\mathbf{B}^{-1}$ with the $\mathbf{c}_i$ ranging over all members of $SU(n_i)$. Additional restrictions on the $\mathbf{c}_i$ may mean that no $\mathbf{r}$ can be found which satisfies these restrictions. (This occurs sometimes for $\mathcal{L} = B_\ell, D_\ell$ or any of the exceptional Lie algebras). $\mathbf{c}$ can however always be expressed in the form

$$\mathbf{c} = \prod_i \exp(a_i \, \mathbf{b}_i) , \qquad (3.38)$$

where the $\mathbf{b}_i$ are generators of $\mathcal{L}_c$. Thus it is useful to know the form of $T(\exp(a_i \, \mathbf{b}_i)) = \exp(\mathrm{ad}(a_i \, \mathbf{b}_i))$. Since

$$\exp(\mathrm{ad}(a_i \, \mathbf{b}_i))\mathbf{x} = \mathbf{x} + a_i[\mathbf{b}_i, \mathbf{x}] + \frac{a_i^{\,2}}{2!}[\mathbf{b}_i, [\mathbf{b}_i, \mathbf{x}]] \; \ldots . \qquad (3.39)$$

by definition, the following results are obtained:-

i) $(\exp \mathrm{ad}\, \lambda_\alpha i\mathbf{h}_\alpha) i\mathbf{h}_\beta = i\mathbf{h}_\beta$,

ii) $(\exp \mathrm{ad}\, \lambda_\alpha i\mathbf{h}_\alpha)(\mathbf{e}_\beta+\mathbf{f}_\beta) = (\mathbf{e}_\beta+\mathbf{f}_\beta)\cos(\beta(\lambda_\alpha \mathbf{h}_\alpha)) + i(\mathbf{e}_\beta-\mathbf{f}_\beta)\sin(\beta(\lambda_\alpha \mathbf{h}_\alpha))$,

iii) $(\exp \mathrm{ad}\, \lambda_\alpha i\mathbf{h}_\alpha) i(\mathbf{e}_\beta-\mathbf{f}_\beta) = i(\mathbf{e}_\beta-\mathbf{f}_\beta)\cos(\beta(\lambda_\alpha \mathbf{h}_\alpha)) - (\mathbf{e}_\beta+\mathbf{f}_\beta)\sin\beta(\lambda_\alpha \mathbf{h}_\alpha))$

iv) $(\exp \mathrm{ad}\, \lambda_\alpha(\mathbf{e}_\alpha+\mathbf{f}_\alpha)) i\mathbf{h}_\beta = i\mathbf{h}_\beta + \dfrac{\alpha(\mathbf{h}_\beta)}{\alpha(\mathbf{h}_\alpha)} \Big(i\mathbf{h}_\alpha(\cos(2\lambda_\alpha\, \alpha(\mathbf{h}_\alpha))^{\frac{1}{2}} - 1)$
$\qquad + \, i(\mathbf{e}_\alpha-\mathbf{f}_\alpha)\left(\dfrac{\lambda_\alpha}{2}\right)^{\frac{1}{2}} \sin(2\lambda_\alpha\, \alpha(\mathbf{h}_\alpha))^{\frac{1}{2}}\Big)$,

v) $(\exp \mathrm{ad}\, \lambda_\alpha(\mathbf{e}_\alpha+\mathbf{f}_\alpha))(\mathbf{e}_\beta+\mathbf{f}_\beta) = A(\mathbf{e}_\beta+\mathbf{f}_\beta) + B(\mathbf{e}_{\alpha+\beta}+\mathbf{f}_{\alpha+\beta})$
$\quad + C(\mathbf{e}_{-\alpha+\beta}+\mathbf{f}_{-\alpha+\beta}) + D(\mathbf{e}_{2\alpha+\beta}+\mathbf{f}_{2\alpha+\beta}) + E(\mathbf{e}_{-2\alpha+\beta}+\mathbf{f}_{-2\alpha+\beta}) + F(\mathbf{e}_{3\alpha+\beta}+\mathbf{f}_{3\alpha+\beta})$
$\quad + G(\mathbf{e}_{-3\alpha+\beta}+\mathbf{f}_{-3\alpha+\beta})$,

vi) $(\exp \mathrm{ad}\, \lambda_\alpha(\mathbf{e}_\alpha+\mathbf{f}_\alpha)) i(\mathbf{e}_\beta-\mathbf{f}_\beta) = Ai(\mathbf{e}_\beta-\mathbf{f}_\beta) + Bi(\mathbf{e}_{\alpha+\beta}-\mathbf{f}_{\alpha+\beta}) + Ci(\mathbf{e}_{-\alpha+\beta}-\mathbf{f}_{-\alpha+\beta})$
$\quad + Di(\mathbf{e}_{2\alpha+\beta}-\mathbf{f}_{2\alpha+\beta}) + Ei(\mathbf{e}_{-2\alpha+\beta}-\mathbf{f}_{-2\alpha+\beta}) + Fi(\mathbf{e}_{3\alpha+\beta}-\mathbf{f}_{3\alpha+\beta}) + Gi(\mathbf{e}_{-3\alpha+\beta}+\mathbf{f}_{-3\alpha+\beta})$,

vii) $(\exp \mathrm{ad}\, \lambda_\alpha\, i(\mathbf{e}_\alpha-\mathbf{f}_\alpha))(\mathbf{e}_\beta+\mathbf{f}_\beta) = A(\mathbf{e}_\beta+\mathbf{f}_\beta) + Bi(\mathbf{e}_{\alpha+\beta}-\mathbf{f}_{\alpha+\beta}) - Ci(\mathbf{e}_{-\alpha+\beta}-\mathbf{f}_{-\alpha+\beta})$
$\quad - D(\mathbf{e}_{2\alpha+\beta}+\mathbf{f}_{2\alpha+\beta}) - E(\mathbf{e}_{-2\alpha+\beta}+\mathbf{f}_{-2\alpha+\beta}) - Fi(\mathbf{e}_{3\alpha+\beta}-\mathbf{f}_{3\alpha+\beta}) + Gi(\mathbf{e}_{-3\alpha+\beta}-\mathbf{f}_{-3\alpha+\beta})$,

viii) $(\exp \mathrm{ad}\, \lambda_\alpha\, i(\mathbf{e}_\alpha-\mathbf{f}_\alpha)) i(\mathbf{e}_\beta-\mathbf{f}_\beta) = Ai(\mathbf{e}_\beta-\mathbf{f}_\beta) - B(\mathbf{e}_{\alpha+\beta}+\mathbf{f}_{\alpha+\beta}) + C(\mathbf{e}_{-\alpha+\beta}+\mathbf{f}_{-\alpha+\beta})$
$\quad - Di(\mathbf{e}_{2\alpha+\beta}-\mathbf{f}_{2\alpha+\beta}) - Ei(\mathbf{e}_{-2\alpha+\beta}-\mathbf{f}_{-2\alpha+\beta}) + F(\mathbf{e}_{3\alpha+\beta}+\mathbf{f}_{3\alpha+\beta}) - G(\mathbf{e}_{-3\alpha+\beta}+\mathbf{f}_{-3\alpha+\beta})$,

where A, B, C, D, E, F, G depend on the α-series of roots containing β. All possible such series are given below (in brackets) together with the non-zero values of A, B, C, D, E, F and G for each series

(β) $$A = 1$$

$(\beta,\ \alpha+\beta)$ $$A = \cosh \lambda_\alpha,\ B = \sinh \lambda_\alpha .$$

$(-\alpha+\beta,\ \beta)$ $$A = \cosh \lambda_\alpha,\ C = \sinh \lambda_\alpha .$$

$(-\alpha+\beta,\ \beta,\ \alpha+\beta)$ $$A = \cosh(2^{\frac{1}{2}}\lambda_\alpha),\ B = C = 2^{-\frac{1}{2}} \sinh(2^{\frac{1}{2}}\lambda_\alpha).$$

$(\beta,\ \alpha+\beta,\ 2\alpha+\beta)$ $$\begin{cases} A = \tfrac{1}{2}+\tfrac{1}{2}\cosh(2^{\frac{1}{2}}\lambda_\alpha),\ B = 2^{-\frac{1}{2}}\sinh(2^{\frac{1}{2}}\lambda_\alpha), \\ D = -\tfrac{1}{2}+\tfrac{1}{2}\cosh(2^{\frac{1}{2}}\lambda_\alpha) . \end{cases}$$

$(-2\alpha+\beta,\ -\alpha+\beta,\ \beta)$ $$\begin{cases} A = \tfrac{1}{2}+\tfrac{1}{2}\cosh(2^{\frac{1}{2}}\lambda_\alpha),\ C = 2^{-\frac{1}{2}}\sinh(2^{\frac{1}{2}}\lambda_\alpha), \\ E = -\tfrac{1}{2}+\tfrac{1}{2}\cosh(2^{\frac{1}{2}}\lambda_\alpha) . \end{cases}$$

$(\beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta)$ $$\begin{cases} A = \frac{(5-\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5+\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ B = \frac{(3\sqrt{5}-5)}{10}\sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(3\sqrt{5}+5)}{10}\sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ D = \frac{\sqrt{5}}{5} (\cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) - \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ F = \frac{(5-\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5+\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \end{cases}$$

$(-\alpha+\beta,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta)$ $$\begin{cases} A = \frac{(5-\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5+\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ B = \frac{(3\sqrt{5}-5)}{10}\sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(3\sqrt{5}+5)}{10}\sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ C = \frac{(5+\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5-\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ D = \frac{\sqrt{5}}{5} (\cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) - \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha)) \end{cases}$$

$(-2\alpha+\beta,\ -\alpha+\beta,\ \beta,\ \alpha+\beta)$ $$\begin{cases} A = \frac{(5-\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5+\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ B = \frac{(5+\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5-\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ C = \frac{(3\sqrt{5}-5)}{10}\sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(3\sqrt{5}-5)}{10}\sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ E = \frac{\sqrt{5}}{5} (\cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) - \frac{\sqrt{5}}{5} \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \end{cases}$$

$(-3\alpha+\beta,\ -2\alpha+\beta,\ -\alpha+\beta,\ \beta)$ $$\begin{cases} A = \frac{(5-\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5+\sqrt{5})}{10} \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ C = \frac{(3\sqrt{5}-5)}{10}\sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(3\sqrt{5}+5)}{10}\sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ E = \frac{\sqrt{5}}{5} \cosh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) - \frac{\sqrt{5}}{5} \cosh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \\ G = \frac{(5-\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1+\sqrt{5})\lambda_\alpha) + \frac{(5+\sqrt{5})}{10} \sinh(\tfrac{1}{2}(1-\sqrt{5})\lambda_\alpha) \end{cases}$$

3.5.5 Embedding when S' and S are both inner

The argument is very similar to that given in [10]. If $\underset{\sim}{c}$ can be expressed in the form (3.37) then the conditions A and B in [10] section 6 hold. More generally, the necessary and sufficient conditions for an embedding are:

A': $T(\exp(\underset{\sim}{\Gamma}(\underset{\sim}{h}')).\underset{\sim}{c})$ is an involutive automorphism of $\mathcal{L}_c$,

B': Trace $\{T(\exp(\underset{\sim}{\Gamma}(\underset{\sim}{h}')).\underset{\sim}{c})\} = -\delta$ (δ is the character of $\mathcal{L}$).

3.5.6 Embeddings when S and S' are both outer

Section 3(b) in [11], sections 3(d), 6(d) and 7(d) of [12] deal with the cases when $\tilde{\mathcal{L}}'$ and $\tilde{\mathcal{L}}$ are classical (A_ℓ or D_ℓ). Thus the remaining cases are when $\tilde{\mathcal{L}}'$ or $\tilde{\mathcal{L}}$ is E_6. Since E_6 is a subalgebra of A_{26} and its outer automorphism can be expressed in terms of the operation of complex conjugation K, the arguments for E_6 are similar to those for A_ℓ.

(a) $\tilde{\mathcal{L}} = E_6$, $\tilde{\mathcal{L}}' = A_{\ell'}$.

S' will be of the form $T(\underset{\sim}{u}').T(\underset{\sim}{d}').K'.\exp(\text{ad } \underset{\sim}{h}')T(\underset{\sim}{u}')^{-1}$, so S'_{ext} will be $T(\underset{\sim}{\Gamma}(\underset{\sim}{u}')).T(\underset{\sim}{\Gamma}(\underset{\sim}{d}')).K'_{ext}.\exp(\text{ad } \underset{\sim}{\Gamma}(\underset{\sim}{h}')).T(\underset{\sim}{\Gamma}(\underset{\sim}{u}'))^{-1}$, where K'_{ext} may be taken as $T(\underset{\sim}{B}).K.T(\underset{\sim}{B}^{-1})$, ($\underset{\sim}{\Gamma}$ being expressed in the form (3.29)). The argument follows that in [11] section 3(b) up to equation (3.20) which becomes

$$\underset{\sim}{c}^*\underset{\sim}{c} = \begin{cases} \underset{\sim}{I}(27) & \text{, if } \mathcal{L}' = SL(\ell'+1, R), \\ \underset{\sim}{\Gamma}(-\underset{\sim}{I}(\ell'+1)) & \text{, if } \mathcal{L}' = Q_{\frac{1}{2}(\ell'+1)}, \end{cases} \tag{3.40}$$

(η being +1, as η must be real and $\eta^{27} = 1$). Taking into account the structure of $\underset{\sim}{c}$ given in (3.32), this condition reduces to:

A: $\underset{\sim}{c}_j''(p_j)^*\underset{\sim}{c}_j''(p_j) = \underset{\sim}{I}(p_j)$, for $\mathcal{L}' = SL(\ell'+1, R)$,

or

A': $\underset{\sim}{c}_j''(p_j)^*\underset{\sim}{c}_j''(p_j) = \sigma_j \underset{\sim}{I}(p_j)$, for $\mathcal{L}' = Q_{\frac{1}{2}(\ell'+1)}$,

where $\underset{\sim}{\Gamma}^j(-\underset{\sim}{I}(\ell'+1)) = \sigma_j \underset{\sim}{I}(n_j)$, σ_j being ± 1.

(b) $\tilde{\mathcal{L}} = E_6$, $\tilde{\mathcal{L}}' = D_{\ell'}$.

The argument follows that given in [12] section 7(d) ($\ell \neq 4$) and section 8(a) ($\ell = 4$) leading to the condition:

A: For every $\underset{\sim}{\Gamma}^j$ appearing in the reduction of $\underset{\sim}{\Gamma}$ (in the form (29)) either

(1) there exists an $\underset{\sim}{s}_j'$ such that $\underset{\sim}{s}_j' \underset{\sim}{\Gamma}^j(\underset{\sim}{b}')^* \underset{\sim}{s}_j'^{-1} = \underset{\sim}{\Gamma}^j(\underset{\sim}{Q}'\underset{\sim}{b}'\underset{\sim}{Q}'^{-1})$ for all $\underset{\sim}{b}'$ in $\mathcal{L}_c'$ or

(2) $\underset{\sim}{\Gamma}^j(\underset{\sim}{b}')$ and $\underset{\sim}{\Gamma}^j(\underset{\sim}{Q}'\underset{\sim}{b}'\underset{\sim}{Q}'^{-1})^* = \underset{\sim}{\Gamma}^{\bar{j}}(\underset{\sim}{b}')^*$ appear the same number of times in the reduction of $\underset{\sim}{\Gamma}$ (i.e. $p_j = p_{\bar{j}*}$).

The extension of $S' = T(\underset{\sim}{Q}')$ is then $T(\underset{\sim}{s})\ K_{ext}$ where

$$K_{ext} = T(\underset{\sim}{B}).K.T(\underset{\sim}{B}^{-1}) \tag{3.41}$$

and $\underset{\sim}{s}$ is of the form

$$\underset{\sim}{s} = \underset{\sim}{B}\Big(\oplus \sum_j (\underset{\sim}{I}(p_j) \otimes \underset{\sim}{s}_j') \oplus \sum_k \underset{\sim}{L}\,(2\, n_k\, p_k)\Big)\underset{\sim}{B}^{-1}, \tag{3.42}$$

where $\underset{\sim}{L}(m)$ is defined for even m in (2.1)

$$\underset{\sim}{L}(m) = \begin{pmatrix} \underset{\sim}{0} & , \underset{\sim}{I}(\tfrac{1}{2}m) \\ \underset{\sim}{I}(\tfrac{1}{2}m), & \underset{\sim}{0} \end{pmatrix}, \tag{2.1}$$

and where the $\underset{\sim}{\Gamma}^j$ satisfying A(1) are listed first and the $\underset{\sim}{\Gamma}^k$ and $\underset{\sim}{\Gamma}^{\bar{k}*}$ satisfying A(2) are listed consecutively in (3.29). The involutive condition then reduces to the form $\underset{\sim}{s}\ \underset{\sim}{c}^* \ (\underset{\sim}{s}\underset{\sim}{c}^*)^* = \underset{\sim}{I}(27)$ ($\eta = +1$ since it must be real and $\eta^{27} = 1$). Considering the structure of $\underset{\sim}{s}$ in (3.42) and $\underset{\sim}{c}$ in (3.32) this becomes:

B: If $\underset{\sim}{\Gamma}^j$ satisfies A(1) then $\underset{\sim}{c}_j''(p_j)^* \underset{\sim}{c}_j''(p_j) = \underset{\sim}{I}(p_j)$ and if $\underset{\sim}{\Gamma}^k$ satisfies A(2) then $\underset{\sim}{c}_k''(p_k)^* \underset{\sim}{c}_{\bar{k}}''(p_k) = \underset{\sim}{c}_{\bar{k}}''(p_k)^* \underset{\sim}{c}_k''(p_k) = \underset{\sim}{I}(p_k)$. The trace condition is:

C: Trace $\{T(\underset{\sim}{s}).K_{ext}.T(\underset{\sim}{c} \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}'))\} = -\delta$.

A, B and C are a set of necessary and sufficient conditions for an embedding.

(c) $\tilde{\mathcal{L}}' = E_6$, $\tilde{\mathcal{L}} = A_\ell$.

Since the outer automorphism of E_6 can be expressed in the same way as that of A_ℓ, the argument is the same as in [11] section 3(b), with $SL(\ell'+1, R)$ replaced by NE_6^3 and $Q_{\frac{1}{2}(\ell'+1)}$ replaced by NE_6^4. Equation (3.28) is used instead of (3.16) and condition A (for $\mathcal{L}' = NE_6^3$) or A' (for $\mathcal{L}' = NE_6^4$) and condition B are a set of necessary and sufficient conditions for an embedding.

(d) $\underline{\tilde{\mathcal{L}}' = E_6,\ \tilde{\mathcal{L}} = D_\ell}$.

The argument given in [12] section 6(d) based on 6(c) and [11] section 5(b), holds for $\tilde{\mathcal{L}}' = E_6$ instead of $\tilde{\mathcal{L}}' = A_{\ell'}$. $SU(\ell'+1)$, $SL(\ell'+1, R)$ and $Q_{\frac{1}{2}(\ell'+1)}$ are everywhere replaced by CE_6, NE_6^3 and NE_6^4 respectively.

3.5.7 Embeddings when S' is inner and S outer

The cases when $\tilde{\mathcal{L}} = A_\ell$ or D_ℓ are dealt with in [11] section 3(c) and [12] section 3(b), the results applying for $\tilde{\mathcal{L}}'$ being classical or exceptional. Thus it only remains to consider the case where $\tilde{\mathcal{L}} = E_6$. The theory is very similar to that given in [11] section 3(c) for $\tilde{\mathcal{L}} = A_\ell$. Since Y must be an outer automorphism it must be of the form $T(\underset{\sim}{y})\, K_{ext}$, where K_{ext} is given in (3.41) and $T(\underset{\sim}{y})\, \underset{\sim}{\Gamma}(\underset{\sim}{b}')^* = \underset{\sim}{\Gamma}(\underset{\sim}{b}')$. This implies the condition

A: $\underset{\sim}{\Gamma}$ must be equivalent to $\underset{\sim}{\Gamma}^*$. Thus every $\underset{\sim}{\Gamma}^j$ occurring in the reduction of $\underset{\sim}{\Gamma}$ as in (3.29) must either (1) be equivalent to $\underset{\sim}{\Gamma}^{j*}$ (i.e. $\underset{\sim}{y}_j'$ exists such that $T(\underset{\sim}{y}_j')\underset{\sim}{\Gamma}^{j*} = \underset{\sim}{\Gamma}^j$), or (2) occur the same number of times as $\underset{\sim}{\Gamma}^{j*}$ in (3.29) (i.e. $p_{j*} = p_j$).

Thus $\underset{\sim}{y}$ is of the form

$$\underset{\sim}{y} = \underset{\sim}{B}\Big(\oplus \sum_j \big(\underset{\sim}{I}(p_j) \otimes \underset{\sim}{y}_j'\big) \oplus \sum_k \underset{\sim}{L}(2n_k p_k)\Big)\underset{\sim}{B}^{-1}, \tag{3.43}$$

assuming that the $\underset{\sim}{\Gamma}^j$ satisfying A(1) are listed first and the $\underset{\sim}{\Gamma}^k$ and $\underset{\sim}{\Gamma}^{k*}$ satisfying A(2) are listed consecutively. By considering the structure of $\underset{\sim}{c}$ and $\underset{\sim}{y}$ given in (3.32) and (3.43) and noting that $\underset{\sim}{y}_j'$ can be chosen such that $\underset{\sim}{y}_j'\underset{\sim}{y}_j'^* = \epsilon_j\, \underset{\sim}{I}(n_j)$, where ϵ_j is +1 for $\underset{\sim}{\Gamma}^j$ real and -1 for $\underset{\sim}{\Gamma}^j$ preudo- real and that $\exp(-2\underset{\sim}{\Gamma}^j(\underset{\sim}{h}')) = \tau_j\, \underset{\sim}{I}(n_j)$, where $\tau_j^{27} = +1$ (τ_j not necessarily real), the involutive condition reduces to

B: (1) If $\underset{\sim}{\Gamma}^j$ satisfies A(1), then τ_j is real and if $\epsilon_j\tau_j = -1$ then p_j must be even. The $\underset{\sim}{c}_j''(p_j)$ must also satisfy $\underset{\sim}{c}_j''(p_j)\underset{\sim}{c}_j''(p_j)^* = \epsilon_j\tau_j\ \underset{\sim}{I}(p_j)$.

(2) If $\underset{\sim}{\Gamma}^k$ and $\underset{\sim}{\Gamma}^{k*}$ satisfy A(2) then the $\underset{\sim}{c}_k''(p_k)$ must satisfy $\underset{\sim}{c}_{k*}''(p_k) = \tau_k{}^*\underset{\sim}{c}_k''(p_k)$.

The trace condition is

C: Trace $\{T(\exp(\underset{\sim}{\Gamma}(\underset{\sim}{h}')).\underset{\sim}{c}.\underset{\sim}{y}).K_{ext}\} = -\delta$.

Conditions A, B and C form a set of necessary and sufficient conditions for an embedding.

3.5.8 Embeddings with S' outer and S inner

This case only applies if $\tilde{\mathcal{L}}' = A_{\ell'}$, $D_{\ell'}$ or E_6. The cases not covered by [10, 11 or 12] are $\tilde{\mathcal{L}}' = E_6$ with $\tilde{\mathcal{L}}$ classical, and $\tilde{\mathcal{L}}' = A_{\ell'}$, $D_{\ell'}$ or E_6 with $\tilde{\mathcal{L}}$ exceptional.

(a) $\tilde{\mathcal{L}}' = E_6$ and $\tilde{\mathcal{L}}$ classical.

This case is very similar to the case where $\tilde{\mathcal{L}}' = A_{\ell'}$ described in [11] section 3(d) and 5(b) and [12] section 6(c). The condition given in these sections apply for $SU(\ell'+1)$, $SL(\ell'+1, R)$ and $Q_{\frac{1}{2}(\ell'+1)}$ replaced everywhere by CE_6, NE_6^3 and NE_6^4 respectively.

(b) $\tilde{\mathcal{L}}' = A_{\ell'}$, E_6 and $\tilde{\mathcal{L}}$ exceptional.

The extension of K' must be of the form $T(\underset{\sim}{y})$ (for $\underset{\sim}{y}$ of the type given in (3.43)) and so a necessary condition is:

A: $\underset{\sim}{\Gamma}$ must be equivalent to $\underset{\sim}{\Gamma}^*$. Thus the $\underset{\sim}{\Gamma}^j$ occurring in the reduction of $\underset{\sim}{\Gamma}$ as in (3.29) must either

(1) be equivalent to $\underset{\sim}{\Gamma}^{j*}$ (i.e. there exists a $\underset{\sim}{y}_j'$ such that $T(\underset{\sim}{y}_j')\underset{\sim}{\Gamma}^{j*} = \underset{\sim}{\Gamma}^j$), or (2) occur the same number of times as $\underset{\sim}{\Gamma}^{j*}$ in (3.29) (i.e. $p_{j*} = p_j$).

The argument follows that given in [11] sections 3(d) and 5(b), the cases with NE_6^3 and NE_6^4 being as for $SL(\ell'+1, R)$ and $Q_{\frac{1}{2}(\ell'+1)}$ respectively, but $\underset{\sim}{c}$ may not necessarily be of the form (3.37), so the involutive condition is:

$$(\underset{\sim}{c}^*\underset{\sim}{y}^*)^2 = \begin{cases} \eta\ \underset{\sim}{I}(r) \quad , & \text{if } \mathcal{L}' = SL(\ell'+1, R) \text{ or } NE_6^3 , \\ \eta\ \underset{\sim}{\Gamma}(-\underset{\sim}{I}(r'), & \text{if } \mathcal{L}' = Q_{\frac{1}{2}(\ell'+1)} \quad \text{or } NE_6^4 , \end{cases} \tag{3.44}$$

where r' and r are the dimensions of $\mathcal{L}_c{}'$ and $\mathcal{L}_c$ respectively. $\eta^r = 1$ and η must be real except when $\tilde{\mathcal{L}} = E_6$. This reduces to the condition

B: (1) If $\underset{\sim}{\Gamma}^j$ satisfies A(1) then $\underset{\sim}{c}_j''(p_j)^2 = \begin{cases} \eta^*\epsilon_j\ \underset{\sim}{I}(p_j), \text{ for } \mathcal{L}' = SL(\ell'+1, R) \text{ or } NE_6^3, \\ \eta^*\epsilon_j\sigma_j\ \underset{\sim}{I}(p_j), \text{ for } \mathcal{L}' = Q_{\frac{1}{2}(\ell'+1)} \text{ or } NE_6^4 , \end{cases}$

and if $\tilde{\mathcal{L}}$ is G_2, F_4, E_7 or E_8 then p_j is even if $\eta = -1$,

or (2) if $\underset{\sim}{\Gamma}^j$ and $\underset{\sim}{\Gamma}^{j^*}$ satisfy A(2) then $\underset{\sim}{c}_{j^*}''(p_j)$ must satisfy

$$\underset{\sim}{c}_{j^*}''(p_j) = \begin{cases} \eta^*\underset{\sim}{c}_j''(p_j)^{-1} \quad , \text{ for } \mathcal{L}' = SL(\ell'+1, R) \text{ or } NE_6^3 , \\ \eta^*\sigma_j\underset{\sim}{c}_j''(p_j)^{-1}, \text{ for } \mathcal{L}' = Q_{\frac{1}{2}(\ell'+1)} \quad \text{or } NE_6^4 . \end{cases}$$

The trace condition is:

C: Trace $\{T(\underset{\sim}{\Gamma}(\underset{\sim}{d}')\ \underset{\sim}{y}\ \underset{\sim}{c} \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}'))\} = -\delta$.

Conditions A, B and C form a set of necessary and sufficient conditions for an embedding.

(c) $\tilde{\mathcal{L}}' = D_{\ell}$, and $\tilde{\mathcal{L}}$ exceptional.

As in [12] section 3(c), a $\underset{\sim}{q}$ must exist such that $T(\underset{\sim}{q})\ \underset{\sim}{\Gamma}(\underset{\sim}{b}') = \underset{\sim}{\Gamma}(T(\underset{\sim}{Q}')\underset{\sim}{b}')$, which implies:

A The $\underset{\sim}{\Gamma}^j$ occurring in the reduction of $\underset{\sim}{\Gamma}$ (29) must satisfy either

(1) there exists a $\underset{\sim}{q}_j{}'$ belonging to $\mathcal{L}_c$ such that $T(\underset{\sim}{q}_j{}')\underset{\sim}{\Gamma}^j(\underset{\sim}{b}') = \underset{\sim}{\Gamma}^j(T(\underset{\sim}{Q}')\underset{\sim}{b}')$ for all $\underset{\sim}{b}'$ in $\mathcal{L}_c$,

or (2) $\underset{\sim}{\Gamma}^j(\underset{\sim}{b}')$ and its conjugate, $\underset{\sim}{\Gamma}^{\bar{j}}(\underset{\sim}{b}') = \underset{\sim}{\Gamma}^j(T(\underset{\sim}{Q}')\underset{\sim}{b}')$ must occur the same number of times (i.e. $p_j = p_{\bar{j}}$).

Thus the extension of $T(\underset{\sim}{Q}')$ is $T(\underset{\sim}{q})$, where

$$\underset{\sim}{q} = \underset{\sim}{B}(\oplus \sum_j (\underset{\sim}{I}(p_j) \otimes \underset{\sim}{q}_j{}') + \sum_k \underset{\sim}{L}(2n_k p_k))\underset{\sim}{B}^{-1} . \tag{3.45}$$

The argument follows that given in [11] section 3(c), 4(b) and 7(c). The involutive condition is $(\underset{\sim}{q}\ \underset{\sim}{c})^2 = \eta \underset{\sim}{I}(r)$ where $\eta^r = 1$. Thus η must be real except when $\tilde{\mathcal{L}} = E_6$. Taking into account the structures of $\underset{\sim}{c}$ and $\underset{\sim}{q}$ given in (3.32) and (3.45), this reduces to the conditions:

B: (1) If $\underset{\sim}{\Gamma}^j$ satisfies A(1) then $\underset{\sim}{c}_j''(p_j)^2 = \eta\ \underset{\sim}{I}(p_j)$ or

(2) If $\underset{\sim}{\Gamma}^k$ satisfies A(2) then $\underset{\sim}{c}_k''(p_k)\underset{\sim}{c}_{\bar{k}}''(p_k) = \eta\ \underset{\sim}{I}(p_k)$.

The trace condition is then

C: Trace $\{T(\underset{\sim}{q}\ \underset{\sim}{c} \exp \underset{\sim}{\Gamma}(\underset{\sim}{h}'))\} = -\delta$.

A, B and C are a set of necessary and sufficient conditions for an embedding.

3.5.9 Summary of procedure for constructing an embedding of $\mathcal{L}'$ in $\mathcal{L}$

($\mathcal{L}'$ and $\mathcal{L}$ are two real Lie algebras not both classical)

1) Use the branching process to find the reduction of all non-conjugate embeddings of $\mathcal{L}_c'$ in $\mathcal{L}_c$. Work through each of these separately.
2) Find an explicit form of $\underset{\sim}{\Gamma}$ (i.e. $\underset{\sim}{\Gamma}(\underset{\sim}{b}_i') = \lambda_{ik}\underset{\sim}{b}_k$). This will usually involve finding $\underset{\sim}{B}$.
3) Find the form of the centralizer elements and the set of $\mathfrak{g}$ if $\underset{\sim}{c}$ can be expressed as $\underset{\sim}{r}(\exp \underset{\sim}{\mathfrak{g}})\underset{\sim}{r}^{-1}$.
4) Choose an S' which generates $\mathcal{L}'$ from $\mathcal{L}_c'$ ($\mathcal{L}' = \sqrt{S'}\ \mathcal{L}_c$) .
5) If S' is an outer automorphism, find its extension (i.e. find which ever of $\underset{\sim}{y}$, $\underset{\sim}{s}$ or $\underset{\sim}{q}$ is appropriate) and calculate ϵ_j or σ_j if applicable.
6) See if the set of necessary and sufficient conditions (given in sections 3.5.5, 3.5.6, 3.5.7 or 3.5.8) for this case are satisfied for any $\underset{\sim}{c}$ in the centralizer and any allowed value of η (if it appears in the conditions).
7) If the conditions in (6) are satisfied then $\mathcal{L}'$ is a subalgebra of $\mathcal{L} = \sqrt{S'_{ext}}\ Y\ \mathcal{L}_c$ and the generators of $\mathcal{L}'$ are the same linear combinations of $\mathcal{L}$ as the generators of $\mathcal{L}_c'$ are of $\mathcal{L}_c$. If the conditions in (6) can not be simultaneously satisfied then $\mathcal{L}'$ is not a subalgebra of $\mathcal{L}$ for any representation conjugate to $\underset{\sim}{\Gamma}$.

3.6. Examples

3.6.1 Subalgebras of real forms of G_2

As will be seen from figure3.1, there are six non-conjugate semi-simple subalgebras of G_2, four of these are non-conjugate embeddings of A_1, and there is embedding of A_2 and of $A_1 \oplus A_1$. We will examine them in the reverse order.

(a) $\tilde{\mathcal{L}}' = A_1 \oplus A_1$

$A_1 \oplus A_1$ is a subalgebra of G_2 with the embedding representation $\underset{\sim}{\Gamma} = \underset{\sim}{B}\{((1) \otimes (1)) \oplus ((0) \otimes (2))\}\underset{\sim}{B}^{-1}$, which can be realised as

$$\left.\begin{aligned}
\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) &= \underset{\sim}{B}(-\underset{\sim}{I}_{1,1} - \underset{\sim}{I}_{2,2} + \underset{\sim}{I}_{3,3} + \underset{\sim}{I}_{4,4})\underset{\sim}{B}^{-1},\\
\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') &= \underset{\sim}{B}(-\underset{\sim}{I}_{1,1} + \underset{\sim}{I}_{2,2} - \underset{\sim}{I}_{3,3} + \underset{\sim}{I}_{4,4} - 2\underset{\sim}{I}_{5,5} + 2\underset{\sim}{I}_{7,7})\underset{\sim}{B}^{-1},\\
\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) &= \underset{\sim}{B}(\underset{\sim}{I}_{1,3} + \underset{\sim}{I}_{2,4})\underset{\sim}{B}^{-1},\\
\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') &= \underset{\sim}{B}(\underset{\sim}{I}_{1,2} + \underset{\sim}{I}_{3,4} + \sqrt{2}\,\underset{\sim}{I}_{5,6} + \sqrt{2}\,\underset{\sim}{I}_{6,7})\underset{\sim}{B}^{-1}.
\end{aligned}\right\}$$

where $\underset{\sim}{B}$ can be taken to be $\underset{\sim}{B} = \underset{\sim}{I}_{1,1} + \underset{\sim}{I}_{2,2} + \underset{\sim}{I}_{3,5} + \underset{\sim}{I}_{4,6} + \underset{\sim}{I}_{5,7} + \underset{\sim}{I}_{6,3} + \underset{\sim}{I}_{7,4}$ and the explicit form of $\underset{\sim}{\Gamma}$ is then

$$\left.\begin{aligned}
\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) &= -(2\underset{\sim}{h}_1 + \underset{\sim}{h}_2)\ , & \underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') &= -\underset{\sim}{h}_2\\
\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) &= \tfrac{1}{6}\,\underset{\sim}{f}_{11222}\ , & \underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') &= \underset{\sim}{f}_2\ .
\end{aligned}\right\}$$

The centralizer elements are of the form

$$\underset{\sim}{c} = \underset{\sim}{B}(c_1\,\underset{\sim}{I}(4) \oplus c_2\,\underset{\sim}{I}(3))\underset{\sim}{B}^{-1} = \text{diagonal } (c_1, c_1, c_2, c_2, c_2, c_1, c_1).$$

Thus $\underset{\sim}{c} = \exp \underset{\sim}{h}$ for $\underset{\sim}{h} = -n\pi i \underset{\sim}{h}_2$ $(n = 0, 1)$, $c_2 = 1$ and $c_1 = \pm 1$.
$SU(2) \otimes SU(2)$ is generated by $S' = T(\exp \underset{\sim}{0})$, the involutive condition is satisfied for both values of n and the trace condition gives an embedding in CG_2 (n = 0 and trace = 14) and in NG_2 (n = 1, trace = -2). $SU(1,1) \otimes SU(2)$ is generated by $T(\exp i\,\frac{\pi}{2}\,(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}))$ and thus $S'_{ext}\ Y = (T(\exp i\pi(\underset{\sim}{h}_1 + (n+\frac{1}{2})\underset{\sim}{h}_2))$ does not satisfy the involutive condition for either value of n. $S'_{ext}\ Y = T(\exp -i\pi(n+\frac{1}{2})\underset{\sim}{h}_2)$ for $SU(2) \otimes SU(1,1)$ and again the involutive condition cannot be satisfied. $SU(1,1) \otimes SU(1,1)$ is generated by $S' = T(\exp(\frac{i\pi}{2}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I} + \underset{\sim}{I} \otimes \underset{\sim}{h}_1'))$, so that $S'_{ext}\ Y$ is $T(\exp - i\pi(\underset{\sim}{h}_1 + (n+1)\underset{\sim}{h}_2))$,

which satisfies the involutive condition for n = 0 and 1. The trace condition gives an embedding in NG_2 (n = 0,1, trace = -2). SL(2, C) is generated by Z', but there is no extension of Z' for this representation (see section 2.4.2), so there is no embedding.

Thus only SU(2) ⊗ SU(2) and SU(1,1) ⊗ SU(1,1) can be embedded in real forms of G_2. The generators of SU(2) ⊗ SU(2), when embedded in CG_2 or NG_2 are $-i\,(2\underset{\sim}{h}_1 + \underset{\sim}{h}_2)$, $-i\underset{\sim}{h}_2$, $\frac{1}{6}(\underset{\sim}{e}_{11222} + \underset{\sim}{f}_{11222})$, $\frac{i}{6}(\underset{\sim}{f}_{11222} - \underset{\sim}{e}_{11222})$, $(\underset{\sim}{e}_2 + \underset{\sim}{f}_2)$, $i(\underset{\sim}{f}_2 - \underset{\sim}{e}_2)$ and the generators of SU(1,1) ⊗ SU(1,1) in NG_2 are $-i(2\underset{\sim}{h}_1 + \underset{\sim}{h}_2)$, $-i\underset{\sim}{h}_2$, $\frac{i}{6}(\underset{\sim}{e}_{11222} + \underset{\sim}{f}_{11222})$, $\frac{1}{6}(\underset{\sim}{e}_{11222} - \underset{\sim}{f}_{11222})$, $i(\underset{\sim}{e}_2 + \underset{\sim}{f}_2)$, $(\underset{\sim}{e}_2 - \underset{\sim}{f}_2)$.

(b) $\underline{\mathcal{L}' = A_2}$

A_2 is a subalgebra of G_2 for which the form of the embedding representation is $\underset{\sim}{B}((1,0) + (0,1) + (0,0))\underset{\sim}{B}^{-1}$, which can be realised as:

$$\left.\begin{aligned}
\underset{\sim}{\Gamma}(\underset{\sim}{h}_1') &= \underset{\sim}{B}(-\underset{\sim}{I}_{1,1} + \underset{\sim}{I}_{2,2} + \underset{\sim}{I}_{4,4} - \underset{\sim}{I}_{5,5})\underset{\sim}{B}^{-1}, \\
\underset{\sim}{\Gamma}(\underset{\sim}{h}_2') &= \underset{\sim}{B}(-\underset{\sim}{I}_{2,2} + \underset{\sim}{I}_{3,3} + \underset{\sim}{I}_{5,5} - \underset{\sim}{I}_{6,6})\underset{\sim}{B}^{-1}, \\
\underset{\sim}{\Gamma}(\underset{\sim}{e}_1') &= \underset{\sim}{B}(\underset{\sim}{I}_{1,2} - \underset{\sim}{I}_{5,4})\underset{\sim}{B}^{-1}, \\
\underset{\sim}{\Gamma}(\underset{\sim}{e}_2') &= \underset{\sim}{B}(\underset{\sim}{I}_{2,3} - \underset{\sim}{I}_{6,5})\underset{\sim}{B}^{-1},
\end{aligned}\right\}$$

where $\underset{\sim}{B}$ can be taken to be $\underset{\sim}{B} = +\underset{\sim}{I}_{1,6} + \underset{\sim}{I}_{2,1} + \underset{\sim}{I}_{3,2} + \underset{\sim}{I}_{4,7} + \underset{\sim}{I}_{5,5} - \underset{\sim}{I}_{6,4} + \underset{\sim}{I}_{7,3}$, and the explicit form of $\underset{\sim}{\Gamma}$ is then

$$\left.\begin{aligned}
\underset{\sim}{\Gamma}(\underset{\sim}{h}_1') &= -\underset{\sim}{h}_1 \ , \quad \underset{\sim}{\Gamma}(\underset{\sim}{h}_2') = -(\underset{\sim}{h}_1 + \underset{\sim}{h}_2), \\
\underset{\sim}{\Gamma}(\underset{\sim}{e}_1') &= +\underset{\sim}{f}_1 \ , \quad \underset{\sim}{\Gamma}(\underset{\sim}{e}_2') = \tfrac{1}{6}\underset{\sim}{f}_{1222} \ .
\end{aligned}\right\}$$

The centralizer elements are of the form

$$\underset{\sim}{c} = \underset{\sim}{B}(c_1\ \underset{\sim}{I}(3) \oplus c_2\ \underset{\sim}{I}(3) \oplus c_3\ \underset{\sim}{I}(1))\underset{\sim}{B}^{-1} = \text{diagonal } (c_2, c_1, c_1, c_3, c_2, c_2, c_1).$$

Thus $\underset{\sim}{c} = \exp \underset{\sim}{h}$, where $\underset{\sim}{h} = \frac{2}{3}n\pi i\ \underset{\sim}{h}_2$, $c_3 = 1$, and $c_1 = c_2{}^* = \exp(\frac{2n\pi i}{3})$.
SU(3) and SU(2,1) are generated by T(exp $\underset{\sim}{h}'$) with $\underset{\sim}{h}' = 0$ and $\frac{i\pi}{3}(2\underset{\sim}{h}_1' + \underset{\sim}{h}_2')$, so that S'_{ext} Y is T(exp $\frac{2n\pi i}{3}\ \underset{\sim}{h}_2$) and T(exp $\pi i(\underset{\sim}{h}_1 + \frac{1}{3}\underset{\sim}{h}_2(1+2n))$) respectively. The involutive condition is only satisfied for n = 0 for $\mathcal{L}'$ = SU(3) and n = 1 for $\mathcal{L}'$ = SU(2,1) and the trace condition allows embeddings of SU(3) in CG_2 (trace = 14) and SU(2,1) in NG_2 (trace = -2). SL(3, R) is generated by T($\underset{\sim}{d}'$).K.

Since in A_2 (0,1) is the complex conjugate of (1,0) it follows that $\underset{\sim}{y} = \underset{\sim}{B}(\underset{\sim}{L}(6) + \underset{\sim}{I}(1))\underset{\sim}{B}^{-1}$, and from [11] section 2

$$\underset{\sim}{\Gamma}(\underset{\sim}{d}') = \underset{\sim}{B}(\underset{\sim}{I}_{1,3} - \underset{\sim}{I}_{2,2} + \underset{\sim}{I}_{3,1} + \underset{\sim}{I}_{4,6} - \underset{\sim}{I}_{5,5} + \underset{\sim}{I}_{6,4} + \underset{\sim}{I}_{7,7})\underset{\sim}{B}^{-1} .$$

Conditions A and B are satisfied, and $\underset{\sim}{\Gamma}(\underset{\sim}{d}')\underset{\sim}{y}.\underset{\sim}{c}$ is $\underset{\sim}{B}(c_1(\underset{\sim}{I}_{1,3} - \underset{\sim}{I}_{2,2} + \underset{\sim}{I}_{3,1}) + c_2(\underset{\sim}{I}_{4,6} - \underset{\sim}{I}_{5,5} + \underset{\sim}{I}_{6,4}) + \underset{\sim}{I}_{7,7})\underset{\sim}{B}^{-1}$, so condition C is satisfied for an embedding of SL(3, R) in NG_2 (trace = −2).

Thus the only embeddings of real forms of A_2 in real forms of G_2 are SU(3) in CG_2, SU(2,1) in NG_2 and SL(3, R) in NG_2. The generators of $\underset{\sim}{\Gamma}(\mathcal{L}')$ are $\sqrt{S'_{ext}\ Y}\ \underset{\sim}{\Gamma}(\underset{\sim}{g}_i')$ where $\mathcal{L}' = \sqrt{S}\ \mathcal{L}_c'$ and the $\underset{\sim}{g}_i'$ are generators of $\mathcal{L}_c'$.

(c) $\underline{\widetilde{\mathcal{L}}' = A_1}$

There are four non-conjugate embeddings of A_1 in G_2 namely corresponding to representations which reduce to (6), (2) + (2) + (0), (2) + (1) + (1) and (1) + (1) + (0) + (0) + (0). They will be examined in this order.

(i) The representation (6) has generators

$$\left.\begin{aligned} \underset{\sim}{\Gamma}(\underset{\sim}{h}_1') &= -(6\underset{\sim}{I}_{1,1} + 4\underset{\sim}{I}_{2,2} + 2\underset{\sim}{I}_{3,3}) + 2\underset{\sim}{I}_{5,5} + 4\underset{\sim}{I}_{6,6} + 6\underset{\sim}{I}_{7,7} , \\ \underset{\sim}{\Gamma}(\underset{\sim}{e}_1') &= \sqrt{6}\ \underset{\sim}{I}_{1,2} + \sqrt{10}\ \underset{\sim}{I}_{2,3} + \sqrt{12}\ \underset{\sim}{I}_{3,4} + \sqrt{12}\ \underset{\sim}{I}_{4,5} + \sqrt{10}\ \underset{\sim}{I}_{5,6} + \sqrt{6}\ \underset{\sim}{I}_{6,7} . \end{aligned}\right\}$$

This representation can be embedded directly in G_2 (i.e. $\underset{\sim}{B} = \underset{\sim}{I}$), the explicit form for $\underset{\sim}{\Gamma}$ being

$$\left.\begin{aligned} \underset{\sim}{\Gamma}(\underset{\sim}{h}_1') &= 6\underset{\sim}{h}_2 + 10\underset{\sim}{h}_1 , \\ \underset{\sim}{\Gamma}(\underset{\sim}{e}_1') &= \sqrt{2}\ \underset{\sim}{f}_2 + \sqrt{10}\ \underset{\sim}{e}_1 . \end{aligned}\right\}$$

The centralizer is the identity (as $\underset{\sim}{\Gamma}$ is irreducible). SU(2) and SU(1,1) are generated by $T(\exp \underset{\sim}{h}')$ for $\underset{\sim}{h}' = \underset{\sim}{0}$ and $\frac{1}{2}\pi\underset{\sim}{h}_1'$ respectively. Thus S'_{ext} Y is $T(\exp \underset{\sim}{0})$ or $T(\exp i\pi(3\underset{\sim}{h}_2 + 5\underset{\sim}{h}_1))$, both of which satisfy the involutive condition. The trace condition allows an embedding of SU(2) in CG_2 and SU(1,1) in NG_2.

(ii) With the representation (2) + (2) + (0), A_1 can be embedded in G_2 in the form

$$\left.\begin{aligned}\Gamma(h_1') &= 2B(-I_{1,1} + I_{3,3} - I_{4,4} + I_{6,6})B^{-1}\ ,\\ \Gamma(e_1') &= \sqrt{2}\ B(I_{1,2} + I_{2,3} + I_{4,5} + I_{5,6})B^{-1}\ ,\end{aligned}\right\}$$

where $B = (I_{1,2} + I_{2,4} + I_{3,6} - I_{4,7} + I_{5,1} + I_{6,3} + I_{7,5})$. Thus $\Gamma(h_1') = 2h_1$ and $\Gamma(e_1') = \frac{3}{\sqrt{2}}(-f_{1222} + e_{1222})$. The centralizer elements are of the form $c = B(c_1''(2) \otimes I(3) \oplus c_2\ I(1))B^{-1}$. c must be a member of CG_2, which implies that the off diagonal elements are zero. Thus $c = \mathrm{diag}(c_{11}, c_{22}, c_{22}, c_2, c_{11}, c_{11}, c_{22}) = \exp h$ for $h = \frac{2n\pi i}{3} h_2$, and hence $c_2 = 1$, $c_{11} = c_{22}^* = \exp \frac{+2n\pi i}{3}$. Thus S'_{ext} Y is $T(\exp \frac{2n\pi i}{3} h_2)$ or $T(\exp i\pi(5h_1 + (3 + \frac{2n}{3})h_2))$. The involutive condition is only satisfied for $n = 3$ in both cases, and the trace condition gives an embedding of SU(2) in CG_2 and SU(1,1) in NG_2.

(iii) The representation (2) + (1) + (1) can be embedded in the form

$$\left.\begin{aligned}\Gamma(h_1') &= B(2(-I_{1,1} + I_{3,3}) - I_{4,4} + I_{5,5} - I_{6,6} + I_{7,7})B^{-1}\ ,\\ \Gamma(e_1') &= B(\sqrt{2}\ (I_{1,2} + I_{2,3}) + I_{4,5} + I_{6,7})B^{-1}\ ,\end{aligned}\right\}$$

for $B = \begin{pmatrix} 0 & I(2) & 0 \\ I(3) & 0 & 0 \\ 0 & 0 & I(2) \end{pmatrix}$.

Thus $\Gamma(h_1') = h_2$ and $\Gamma(e_1') = e_2$, and the centralizer elements are of the form $c = B(c_1''(2) \otimes I(2) \oplus c_2\ I(3))B^{-1}$, which can be diagonalised to the form $c = r(\exp g)r^{-1}$ with $g = n\pi i\ h_2$ $(n = 0,1)$. The involutive condition is satisfied for $S' = T(0)$ but not for $S' = T(\exp \frac{1}{2}\pi i\ h_1')$. The trace condition allows an embedding of SU(2) in CG_2 ($n = 0$, trace = 14) and in NG_2 ($n = 1$, trace = -2).

(iv) The representation (1) + (1) + 3(0) of A_1 can be embedded in the form

$$\left.\begin{aligned}\Gamma(h_1') &= B(-I_{1,1} + I_{2,2} - I_{3,3} + I_{4,4})B^{-1}\ ,\\ \Gamma(e_1') &= B(I_{1,2} + I_{3,4})B^{-1}\ ,\end{aligned}\right\}$$

with $\qquad B = I_{1,5} + I_{2,1} + I_{3,2} + I_{4,6} + I_{5,3} + I_{6,4} + I_{7,7}$.

Thus $\Gamma(h_1') = h_1$ and $\Gamma(e_1') = e_1$. The centralizer elements are of the form $B(c_1''(2) \otimes I(2) \oplus c_2'(3))B^{-1}$, which cannot be put in the form (3.44) but can be expressed in the form

Table 3.1 Subalgebras of real forms of G_2.

Complex Lie algebra	Reduction of representation	Subalgebras	
		CG_2	NG_2
$A_1 \oplus A_1$	$(1) \otimes (1) + (0) \otimes (2)$	$SU(2) \otimes SU(2)$	$SU(2) \otimes SU(2)$ $SU(1,1) \otimes SU(1,1)$
A_2	$(1,0) + (0,1) + (0,0)$	$SU(3)$	$SU(2,1)$ $SL(3,R)$
A_1	(6)	$SU(2)$	$SU(1,1)$
	$(2) + (2) + (0)$	$SU(2)$	$SU(1,1)$
	$(2) + (1) + (1)$	$SU(2)$	—
	$(1) + (1) + 3(0)$	$SU(2)$	—

$$c = \exp \lambda_1\, i(h_1 + \tfrac{2}{3} h_2) \cdot \exp \lambda_2(e_{122} + f_{122}) \cdot \exp \lambda_3\, i(e_{122} - f_{122})$$

For SU(2), $S'_{ext}\ Y = T(c)$, the involutive condition is only satisfied for $\lambda_2 = \lambda_3 = 0$ and $\lambda_1 = 6\pi ni$, and then $c = I$. The trace condition allows for an embedding of SU(2) in CG_2. $S'_{ext}\ Y = T((\exp \frac{i\pi}{2} h_2)c)$ for SU(1,1), but the involutive condition cannot be satisfied for any value of c so there is no embedding of SU(1,1).

The semi-simple subalgebras of real forms of G_2 are summarized in table 3.1.

3.6.2 Subalgebras of real forms of F_4

The subalgebras of F_4 are too numerous to consider in their entirety here so we will only examine embeddings of real forms of B_4 ((0,0,0,1) + (1,0,0,0)), A_1 ((16) + (9)) and all embeddings of SL(2, C).

(a) $\tilde{\mathcal{L}}' = B_4$

B_4 (0,0,0,1) + (1,0,0,0) can be embedded with the explicit form for $\Gamma = B((0,0,0,1) \oplus (1,0,0,0) \oplus (0,0,0,0))B^{-1}$ being

$$\left.\begin{array}{ll} \Gamma(h_1') = h_2 & , \quad \Gamma(e_1') = -f_2 , \\ \Gamma(h_2') = h_1 & , \quad \Gamma(e_2') = -f_1 , \\ \Gamma(h_3') = h_2 + h_3 & , \quad \Gamma(e_3') = -f_{233} , \\ \Gamma(h_4') = h_4 & , \quad \Gamma(e_4') = -f_4 . \end{array}\right\}$$

The centralizer elements are of the form

$$c = B(c_1'\ I(16) \oplus c_2'\ I(9) + c_3\ I(1))B^{-1} ,$$

which is a member of CF_4 if $c_2 = c_3 = 1$ and $c_1 = \pm 1$. Thus $c = \exp n\pi i\, h_1$ where $n = 0,1$.

The real forms of B_4 are SO(9 - 2p, 2p) for p = 0,...,4, which are all generated by inner automorphisms, and $S'_{ext}\ Y$ is $T(\exp h)$ for $h = n\pi i\, h_1$ (p = 0), $\pi i((4+n)h_1 + 8h_2 + 6h_3 + \frac{7}{2} h_4)$ (p=1), $\pi i((4+n)h_1 + 7h_2 + 5h_3 + 3h_4)$ (p=2), $\pi i((3+n)h_1 + 6h_2 + 4h_3 + 5h_4)$ (p=3), $\pi i((2+n)h_1 + 4h_2 + 3h_3 + 2h_4)$ (p=4). The involutive condition on $T(\exp \pi i \sum_{i=1}^{4} \lambda_i h_i)$ is only satisfied if all the

λ_i are integers. Thus it is not satisfied for p = 1,3. The trace condition gives embedding of SO(9) in CF_4 (n = 0, trace = 52) and NF_4^1 (n = 1, trace = -4), SO(5,4) in NF_4^1 (n = 0,1, trace = -4) and SO(1,8) in NF_4^1 (n = 1, trace = -4) and NF_4^2 (n = 0, trace = 20).

(b) $\tilde{\mathcal{L}}' = A_1$

The representation (16) + (8) of A_1 can be embedded in F_4 with the explicit form for $\underset{\sim}{\Gamma} = \underset{\sim}{B}((16)+(8))\underset{\sim}{B}^{-1}$ such that

$$\left.\begin{aligned} \underset{\sim}{\Gamma}(\underset{\sim}{h}_1') &= 6\underset{\sim}{h}_1 + 4\underset{\sim}{h}_2 + 12\underset{\sim}{h}_3 + 8\underset{\sim}{h}_4, \\ \underset{\sim}{\Gamma}(\underset{\sim}{f}_1') &= 4\ \underset{\sim}{f}_{123} - \sqrt{30}\,\underset{\sim}{e}_{1234} - \frac{21}{2}\underset{\sim}{e}_{233} \\ &\quad + \sqrt{\frac{165}{92}}\ \underset{\sim}{f}_{1222333344}\,. \end{aligned}\right\}$$

The centralizer elements are of the form $\underset{\sim}{c} = \underset{\sim}{B}(c_1\ \underset{\sim}{I}(17) \oplus c_2\ \underset{\sim}{I}(9))\underset{\sim}{B}^{-1}$, but both c_1 and c_2 must be 1(so that $\underset{\sim}{c} = \underset{\sim}{I}$). The involutive condition is satisfied for $S'_{ext}\ Y$ being both $T(\underset{\sim}{0})$ and $T(\exp(3\underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 + 12\underset{\sim}{h}_3 + 8\underset{\sim}{h}_4)$ (i.e. for SU(2) and SU(1,1)), and the trace condition allows the embeddings SU(2) in CF_4 and SU(1,1) in NF_4^1.

(c) $\mathcal{L}' = SL(2, C)$

Any representation which provides for an embedding of SL(2, C) must reduce to a form which only contains terms $\underset{\sim}{\Gamma}^j \otimes \underset{\sim}{\Gamma}^j$ and $(\underset{\sim}{\Gamma}^j \otimes \underset{\sim}{\Gamma}^k)\oplus(\underset{\sim}{\Gamma}^k \otimes \underset{\sim}{\Gamma}^j)$ (see section 2.4.2). Thus many of the representations of $A_1 \oplus A_1$ may be eliminated, and the only three remaining representations are

$$\left.\begin{aligned} \underset{\sim}{\Gamma}_1 &= 4((0)\otimes(1) + (1)\otimes(0)) + (1)\otimes(1) + 6((0)\otimes(0)), \\ \underset{\sim}{\Gamma}_2 &= ((0)\otimes(2) + (2)\otimes(0)) + 4((1)\otimes(1)) + 4((0)\otimes(0)), \\ \underset{\sim}{\Gamma}_3 &= ((1)\otimes(3) + (3)\otimes(1)) + (2)\otimes(2) + (0)\otimes(0)\,. \end{aligned}\right\}$$

(i) An explicit form for Γ_1 is $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{h}_2$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{e}_2$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') = \underset{\sim}{h}_2 + \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') = \frac{1}{2}\underset{\sim}{e}_{233}$, where $\underset{\sim}{\Gamma} = \underset{\sim}{B}\ \underset{\sim}{\Gamma}_1 \underset{\sim}{B}^{-1}$.

The centralizer cannot be reduced to diagonal form but can be expressed as

$$\underset{\sim}{c} = \exp \lambda(\underset{\sim}{h}_2+2\underset{\sim}{h}_3+2\underset{\sim}{h}_4)\ .\ \prod_\alpha(\exp \lambda_\alpha{}^1(\underset{\sim}{e}_\alpha+\underset{\sim}{f}_\alpha)\ .\ \exp \lambda_\alpha{}^2 i(\underset{\sim}{e}_\alpha-\underset{\sim}{f}_\alpha)\ ,$$

where the $\lambda_\alpha^{\ i}$ are zero except for α being $\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$, $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$. Z'_{ext} can be taken as $T(\underset{\sim}{B}(\underset{\sim}{L}(16) \oplus \underset{\sim}{w}'_{ext}(4) \oplus \underset{\sim}{I}(6))\underset{\sim}{B}^{-1})$, where $\underset{\sim}{w}'_{ext}$ is defined in section 2.4. The involutive condition is however only satisfied for $\lambda_\alpha^{\ 1} = 0$ for all α and $\lambda_\alpha^{\ 2} = 0$ for all α except $\alpha_2+2\alpha_3+2\alpha_4$ and $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$, when it is a multiple of πi. The trace condition allows an embedding of SL(2, C) in NF_4^2 ($\lambda = 0$, trace = 20) and NF_4^1 ($\lambda = \pi i$, trace = -4).

(ii) $\underset{\sim}{\Gamma}_2$ can be embedded in the form $\underset{\sim}{\Gamma} = \underset{\sim}{B}\,\underset{\sim}{\Gamma}_2\,\underset{\sim}{B}^{-1}$ where $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{e}_{12} + \underset{\sim}{e}_{1233}$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') = 2\underset{\sim}{h}_2 + \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') = \underset{\sim}{e}_{123344} + \underset{\sim}{e}_{1222333344}$. Again the centralizer cannot be diagonalized but can be expressed as

$$\underset{\sim}{c} = \prod_\alpha (\exp \lambda_\alpha^{\ 1}(\underset{\sim}{e}_\alpha + \underset{\sim}{f}_\alpha) \,.\, \exp \lambda_\alpha^{\ 2} i(\underset{\sim}{e}_\alpha - \underset{\sim}{f}_\alpha)) \ ,$$

where $\lambda_\alpha^{\ i}$ are zero except for $\alpha = \alpha_2+2\alpha_3+2\alpha_4$ and $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$. Z'_{ext} can be taken as

$$Z'_{ext} = T(\underset{\sim}{B}(\underset{\sim}{L}(6) \oplus \underset{\sim}{I}(4) \otimes \underset{\sim}{w}'_{ext}(4) \oplus \underset{\sim}{I}(4))\underset{\sim}{B}^{-1}) \ .$$

The involutive condition $(Z'_{ext}\ T(\underset{\sim}{c}))^2 = \underset{\sim}{I}$ implies that $\lambda_\alpha^{\ 1} = 0$ for both α and $\lambda_\alpha^{\ 2}$ is a multiple of π.

For $\lambda_\alpha^{\ 2} = 0$ for both α the trace condition allows an embedding of SL(2, C) in NF_4^2 (trace 20), and for $\lambda_\alpha^{\ 2} = \pi$ for both α the trace condition allows an embedding of SL(2, C) in NF_4^1 (trace -4). Thus SL(2, C) can be embedded in both non-compact real forms of F_4.

(iii) The representation $\underset{\sim}{\Gamma}_3$ can be embedded in the form $\underset{\sim}{\Gamma} = \underset{\sim}{B}\,\underset{\sim}{\Gamma}_3\,\underset{\sim}{B}^{-1}$, where $\underset{\sim}{B}$ is such that the explicit form of $\underset{\sim}{\Gamma}$ is

$$\left.\begin{array}{c} \underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{h}_3 - \underset{\sim}{h}_4,\ \underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') = -2\underset{\sim}{h}_1 + \underset{\sim}{h}_3 + \underset{\sim}{h}_4\ , \\ \underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{e}_{1223334} + \underset{\sim}{f}_{12} - \underset{\sim}{f}_{23344}\ , \\ \underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') = -\underset{\sim}{e}_{34} - \underset{\sim}{f}_{1222333344} + \underset{\sim}{e}_{11222333344}\ . \end{array}\right\}$$

The centralizer is diagonal and therefore can be expressed as $\underset{\sim}{c} = \exp 2n\pi i(\underset{\sim}{h}_3 + \underset{\sim}{h}_4)$. Z'_{ext} can be taken as

$$Z'_{ext} = T(\underset{\sim}{B}(\underset{\sim}{W}(8,8) \oplus \underset{\sim}{w}'_{ext}(9) \oplus \underset{\sim}{I}(1)\underset{\sim}{B}^{-1}\ ,$$

where $\underset{\sim}{W}(n_p, n_q)$ is defined in section 2.4. For n = 0 and 1 the involutive condition is satisfied and the trace condition gives an embedding of SL(2, C) in NF_4^1 (trace = -4).

3.6.3 Real forms of E_6 containing SL(2, C)

We shall only consider embeddings of SL(2, C) in real forms of E_6. As for F_4, embedding representations may be eliminated if there is no extension to Z' (see section 2.4.2). The only remaining representations are

$$\underset{\sim}{\Gamma}_1 = 4(((0)\otimes(1)) + (1)\otimes(0)) + (1)\otimes(1) + 7((0)\otimes(0)),$$

$$\underset{\sim}{\Gamma}_2 = (0)\otimes(2) + (2)\otimes(0) + 4((1)\otimes(1)) + 5((0)\otimes(0)),$$

$$\underset{\sim}{\Gamma}_3 = (1)\otimes(3) + (3)\otimes(1) + (2)\otimes(2) + 2(0)\otimes(0),$$

$$\underset{\sim}{\Gamma}_4 = 3((2)\otimes(0) + (0)\otimes(2)) + (2)\otimes(2),$$

$$\underset{\sim}{\Gamma}_5 = (2)\otimes(0) + (0)\otimes(2) + (2)\otimes(1) + (1)\otimes(2) + (1)\otimes(0) + (0)\otimes(1) + (1)\otimes(1) + (0)\otimes(0),$$

$$\underset{\sim}{\Gamma}_6 = (4)\otimes(0) + (0)\otimes(4) + (3)\otimes(3) + (0)\otimes(0).$$

(i) $\underset{\sim}{\Gamma}_1$ can be embedded in the form $\underset{\sim}{\Gamma} = \underset{\sim}{B}\,\underset{\sim}{\Gamma}_1\,\underset{\sim}{B}^{-1}$ with $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{h}_1$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{e}_1$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') = \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') = \underset{\sim}{e}_3$. The centralizer elements cannot be diagonalized, but they are of the form

$$\underset{\sim}{c} = \exp(a_1(\underset{\sim}{h}_1+2\underset{\sim}{h}_2+\underset{\sim}{h}_3) + a_2(\tfrac{1}{2}\underset{\sim}{h}_3+\underset{\sim}{h}_4) + a_3\underset{\sim}{h}_5 + a_4(\tfrac{1}{2}\underset{\sim}{h}_3+\underset{\sim}{h}_6)) \,.\, \prod_{\alpha} \exp(\lambda_\alpha{}^1(\underset{\sim}{e}_\alpha+\underset{\sim}{f}_\alpha)) \,.\, \exp(\lambda_\alpha{}^2 i(\underset{\sim}{e}_\alpha-\underset{\sim}{f}_\alpha)) ,$$

where $\lambda_\alpha{}^i$ is zero except for α being $\alpha_3+\alpha_4+\alpha_6$, $\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$, α_5, $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_6$. Z'_{ext} is given by

$$Z'_{ext} = T(\underset{\sim}{B}(\underset{\sim}{L}(16) \oplus \underset{\sim}{w}'_{ext}(4) \oplus \underset{\sim}{I}(7))\underset{\sim}{B}^{-1}) ,$$

For S inner, the involutive condition implies that $\lambda_\alpha{}^1 = 0$ for all α and $\lambda_\alpha{}^2$ is a multiple of π for all α, and that $a_1 = a_3 = n\pi i$ and $a_2 = a_4 = 2m\pi i$. The trace condition allows for an embedding in NE_6^2 ($n = 0 = \lambda_\alpha{}^2$, trace = 14) and NE_6^1 (n = 1, trace = -2). For S being outer, Y is outer and thus a $\underset{\sim}{y}$ is required such that $T(\underset{\sim}{y})\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}^*$, such a $\underset{\sim}{y}$ being given by $\underset{\sim}{y} = \underset{\sim}{B}(\sum_{i=1}^{9} \underset{\sim}{y}_i \oplus \underset{\sim}{I}(7))\underset{\sim}{B}^{-1}$

for $y_i = M(2)$ when $i = 1,\ldots,8$ and $y_9 = M(4)$, with $M(n)$ given by

$$(M(n))_{pq} = \delta_{p+q,\, n+1} .$$

The involutive condition puts no restriction on the centralizer elements, but the trace condition is only satisfied for NE_6^4 (trace -6). Thus SL(2, C) can be embedded in NE_6^1, NE_6^2 and NE_6^4 with representation $B\,\Gamma_1\,B^{-1}$.

(ii) Γ_2 can be embedded in the form $\Gamma = B\,\Gamma_2\,B^{-1}$ with $\Gamma(h_1' \otimes I) = h_5 + h_6$, $\Gamma(e_1' \otimes I) = e_{3456} + f_{34}$, $\Gamma(I \otimes h_1') = h_5 - h_6$, $\Gamma(I \otimes e_1') = e_{345} - f_{346}$. The centralizer elements are of the form

$$c = \exp(a_1 h_1 + a_2(h_2+2h_4+h_5)) \cdot \prod_\alpha (\exp \lambda_\alpha{}'(e_\alpha + f_\alpha) \cdot \exp \lambda_\alpha{}^2 i(e_\alpha - f_\alpha)) ,$$

where the $\lambda_\alpha{}^i$ are zero except for $\alpha = \alpha_1$ or $\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$, which cannot be diagonalized to the form(3.37). Z'_{ext} can be taken as

$$Z'_{ext} = T(B(L(6) \oplus w'_{ext}(4) \oplus w'_{ext}(4) \oplus w'_{ext}(4) \oplus w'_{ext}(4) \oplus I(5))B^{-1}) .$$

For S inner the involutive condition implies that $\lambda_\alpha{}^i = 0$ for all α and $\lambda_\alpha{}^2$ is a multiple of π and $a_1 = n\pi$ and $a_2 = 2n\pi i$. The trace condition allows for an embedding in NE_6^2 ($n = 0$, trace $= -2$) and in NE_6^1 ($n = 1$, trace $= +14$). If S is outer then y can be taken as $B(\sum_{i=1}^{6} y_i + I(5))B^{-1}$, where y_i is $M(3)$ for $i = 1,2$ and $M(4)$ for $i = 3,\ldots,6$.

The involutive condition puts no restriction on c but the trace condition is only satisfied for NE_6^4 (all values of c give trace -6). Thus SL(2, C) can be embedded in NE_6^1, NE_6^2 and NE_6^4 with representation Γ_2.

(iii) $\Gamma = B\,\Gamma_3\,B^{-1}$ can be embedded such that $\Gamma(h_1' \otimes I) = 2h_3 + h_5 + h_1$, $\Gamma(e_1' \otimes I) = +f_{12346} - f_{23456} - \sqrt{2}\,f_{234} + \sqrt{2}\,e_{1223334456}$, $\Gamma(I \otimes h_1') = 2h_3 - h_5 - h_1 + 2h_6$, $\Gamma(I \otimes e_1') = \sqrt{2}\,f_{123456} - f_{1234} + \sqrt{2}\,e_{12233344566} + f_{2345}$. The centralizer elements can be diagonalized and can be expressed as $c = r(\exp g)r^{-1}$, where g is $\lambda_i(h_1 + 2h_2 - 2h_4 - h_5)$. Z'_{ext} can be taken as

$$Z'_{ext} = T(B(W(8,8) \oplus w'_{ext}(9) \oplus I(2))B^{-1}).$$

For S inner, the involutive condition is satisfied for $\lambda = n\pi$ and the trace condition gives an embedding in NE_6^2 ($n = 0,1$, trace $= -2$).

If S is outer $\underset{\sim}{y}$ can be taken as $\underset{\sim}{B}(\sum_{i=1}^{3} \underset{\sim}{y}_i \oplus \underset{\sim}{I}(2))\underset{\sim}{B}^{-1}$, where $\underset{\sim}{y}_1 = \underset{\sim}{y}_2 = \underset{\sim}{M}(8)$ and $\underset{\sim}{y}_3 = \underset{\sim}{M}(9)$. The involutive condition is satisfied for all λ and the trace condition gives an embedding in NE_6^4 (trace $= -6$ for all values of $\underset{\sim}{c}$).

Thus SL(2, C) can be embedded in NE_6^2 and NE_6^4 with the representation $\underset{\sim}{\Gamma}_3$.

(iv) The embedding with representation $\underset{\sim}{\Gamma} = \underset{\sim}{B}\,\underset{\sim}{\Gamma}_4\,\underset{\sim}{B}^{-1}$ can be realised in the form where $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) = 2\underset{\sim}{h}_1$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{e}_{1236} - \underset{\sim}{f}_{236}$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') = 2\underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{e}_1') = \underset{\sim}{e}_{1223334456} - \underset{\sim}{f}_{122334456}$. The extension Z' may be taken as

$$Z'_{ext} = T(\underset{\sim}{B}(\underset{\sim}{L}(6) \oplus \underset{\sim}{w}'_{ext}(9))\underset{\sim}{B}^{-1}) ,$$

and the centralizer elements can be diagonalized to the form $\underset{\sim}{c} = \underset{\sim}{r}(\exp \underset{\sim}{g})\underset{\sim}{r}^{-1}$, where g is $n\pi i(\underset{\sim}{h}_4 - \underset{\sim}{h}_6) + \lambda i \underset{\sim}{h}_5$. For S inner, the involutive condition is satisfied for $n = 0,1$, $\lambda = m\pi$, and the trace condition is satisfied for embeddings in NE_6^2 ($n = m = 0$, trace $= 14$) and NE_6^1 ($n = 1$ or $m = 1$, trace $= -2$). When S is outer, $\underset{\sim}{y}$ may be taken as $\underset{\sim}{B}(\oplus \sum_{i=1}^{7} \underset{\sim}{y}_i)\underset{\sim}{B}^{-1}$, where $\underset{\sim}{y}_i = \underset{\sim}{M}(3)$ for $i=1,\ldots,6$ and $\underset{\sim}{y}_7 = \underset{\sim}{M}(9)$. The involutive condition is satisfied for all $\underset{\sim}{c}$ and the trace condition is satisfied for an embedding in NE_6^4 (trace $= -6$ for all values of $\underset{\sim}{c}$).

Thus SL(2, C) can be embedded in NE_6^1, NE_6^2 and NE_6^4 with representation $\underset{\sim}{\Gamma}_4$.

(v) $\underset{\sim}{\Gamma} = \underset{\sim}{B}\,\underset{\sim}{\Gamma}_5\,\underset{\sim}{B}^{-1}$ can be embedded when $\underset{\sim}{\Gamma}(\underset{\sim}{h}_1' \otimes \underset{\sim}{I}) = 2\underset{\sim}{h}_6 + \underset{\sim}{h}_3$, $\underset{\sim}{\Gamma}(\underset{\sim}{e}_1' \otimes \underset{\sim}{I}) = \underset{\sim}{e}_{12233344566} + \underset{\sim}{f}_{2345} - \underset{\sim}{f}_{1234}$, $\underset{\sim}{\Gamma}(\underset{\sim}{I} \otimes \underset{\sim}{h}_1') = \underset{\sim}{h}_1 + \underset{\sim}{h}_3 + \underset{\sim}{h}_5$, $\underset{\sim}{\Gamma}(\underset{\sim}{I}' \otimes \underset{\sim}{e}_1') = \underset{\sim}{e}_1 + \underset{\sim}{e}_3 + \underset{\sim}{e}_5$. The centralizer elements are diagonal of the form $\underset{\sim}{c} = \exp \lambda(\underset{\sim}{h}_1 + 2\underset{\sim}{h}_2 - 2\underset{\sim}{h}_4 - \underset{\sim}{h}_5)$, and Z'_{ext} may be taken as

$$Z'_{ext} = T(\underset{\sim}{B}(\underset{\sim}{L}(6) \oplus \underset{\sim}{W}(6,6) \oplus \underset{\sim}{L}(4) \oplus \underset{\sim}{w}'_{ext}(4) \oplus \underset{\sim}{I}(1))\underset{\sim}{B}^{-1}) .$$

For S inner, the involutive condition implies that $\lambda = n\pi$ ($n = 0,1$) and the trace condition is satisfied for an embedding of SL(2, C) in NE_6^2. For S outer, $\underset{\sim}{y}$ can be taken as $\underset{\sim}{B}(\sum_{i=1}^{7} \underset{\sim}{y}_i \oplus \underset{\sim}{I}(1))\underset{\sim}{B}^{-1}$, where $\underset{\sim}{y}_1 = \underset{\sim}{y}_2 = \underset{\sim}{M}(3)$, $\underset{\sim}{y}_3 = \underset{\sim}{y}_4 = \underset{\sim}{M}(6)$, $\underset{\sim}{y}_5 = \underset{\sim}{y}_6 = \underset{\sim}{M}(2)$ and $\underset{\sim}{y}_7 = \underset{\sim}{M}(4)$. The involutive condition is satisfied for all $\underset{\sim}{c}$ and the trace condition for an embedding in NE_6^4 (trace $= -6$ for all values of $\underset{\sim}{c}$).

Thus SL(2, C) can be embedded in NE_6^2 and NE_6^4 with representation Γ_5.

(vi) Γ_6 can be embedded in the form of $\Gamma = B\,\Gamma_6\,B^{-1}$, where $\Gamma(h_1' \otimes I) = 4h_1 + 4h_2 + 2h_3 + h_4 + h_6$, $\Gamma(e_1' \otimes I) = 2e_{123346} - \sqrt{3}\,f_{3,6} - \sqrt{3}\,f_{34}$, $\Gamma(I \otimes h_1') = 4h_2 + 4h_3 + h_4 + h_6$ and $\Gamma(I \otimes e_1') = 2e_{2346} - \sqrt{3}\,f_4 - \sqrt{3}\,f_6$. The centralizer consists of elements of the form $e^{\frac{2\pi ni}{3}} I$, so that T(c) is the identity operation. The extension of Z' is $T(B(L(10) \oplus w'_{ext}(16) \oplus I(1))B^{-1})$ and for S inner the involutive and trace conditions are satisfied for an embedding in NE_6^2 (trace 14). For S outer, y may be taken $B(M(5) \oplus M(5) \oplus M(16) \oplus I(1))B^{-1}$, and the involutive and trace conditions are satisfied for an embedding in NE_6^4 (trace = −6).

Thus SL(2, C) can be embedded in NE_6^2 and NE_6^4 with representation Γ_6.

3.7. Conclusions concerning embeddings of SL(2, C)

In section 2.9, it was shown that the list of real Lie algebras containing SL(2, C) given by Barut and Raczka [1] was incomplete. The results of this indicate that there are many exceptional Lie algebras which contain SL(2, C), which were also not mentioned in [1]. It has also been shown that SL(2, C) may be isomorphic to several non-conjugate subalgebras of a real form of an exceptional Lie algebra. The following list gives all exceptional Lie algebras of rank ≤ 6, which contain subalgebras isomorphic to SL(2, C) (the number of non-conjugate embeddings is given in brackets after each algebra): $NF_4^1(3)$, $NF_4^2(2)$, $NE_6^1(3)$, $NE_6^2(6)$, $NE_6^4(6)$.

CHAPTER 4

EMBEDDINGS OF LIE GROUPS IN LIE GROUPS

4.1 Lie Groups and Lie Algebras.

All locally isomorphic Lie groups have isomorphic Lie algebras, since the generators $\underset{\sim}{a}_i$ of a Lie algebra $\mathcal{L}$ are the infinitesimal generators of its Lie group G. Thus all elements $\underset{\sim}{g} \in G$ in a neighbourhood of the identity can be expressed as

$$\underset{\sim}{g} = \underset{\sim}{I} + t_i \underset{\sim}{a}_i \tag{4.1}$$

More generally any $\underset{\sim}{g} \in G$ can be expressed as

$$\underset{\sim}{g} = \exp (t_i \underset{\sim}{a}_i) \tag{4.2}$$

which becomes (4.1) in the limit as the t_i tend to zero. Thus the infinitesimal elements $\underset{\sim}{a}$ of G (unit elements of $\mathcal{L}$) can be obtained from the elements of G :

$$\underset{\sim}{a} = \text{limit}\left\{\frac{(\underset{\sim}{g} - \underset{\sim}{I})}{|\underset{\sim}{g} - \underset{\sim}{I}|}\right\} \quad \text{as } \underset{\sim}{g} \to \underset{\sim}{I} \text{ along a continuous path .} \tag{4.3}$$

All continuous representations of G (single-valued, multi-valued, faithful or unfaithful) will have isomorphic Lie algebras. There is one simply connected Lie group $\tilde{G}$, (unique up to isomorphism), which has a Lie algebra isomorphic to $\mathcal{L}$. This group is the <u>Universal covering group</u> of all groups locally isomorphic to it, Every group G is a <u>factor group</u> of its universal covering group :

$$G = \tilde{G}/N \quad , \tag{4.4}$$

where N is a normal (invariant) central subgroup of $\tilde{G}$. Hence every representation of $\tilde{G}$ is a representation of G and vice-versa. All representations of $\tilde{G}$ are single valued and all representations of groups with trivial centres are either faithful or trivial.

A linear group is one which has at least one faithful linear representation , i.e. a representation in the form of linear operators (equivalent to a matrix representation) . It is these groups which are physically important , they are each isomorphic to a subgroup of the general linear group $GL(N,C)$, where N is the dimension of the group . All compact universal covering groups are linear , but non-compact universal covering groups are not all linear . Thus it is useful to define a universal linear group $\hat{G}$, as in [36] , unique up to isomorphism , which is a linear group , of which all other locally isomorphic linear groups are factor groups .

From theorem 4 , a faithful linear representation of G provides a faithful linear representation of $\mathcal{L}$ and a faithful linear representation of $\mathcal{L}$ provides a linear (but not necessarily faithful) representation of G . Thus if $\Gamma(\mathcal{L}') \subset \mathcal{L}$, it is not necessary for $G' \subset G$, since Γ may not be a faithful representation of G'. The problem for non-linear groups is mathematically complex , but physically unimportant . Results are therefore obtained for linear semi-simple groups only .

4.2 Embeddings of a linear semi-simple Lie group G' in a linear semi-simple Lie group G .

4.2.1 General Theory

To avoid confusion in this section , G will denote the abstract Lie group and $\mathcal{L}$ its abstract Lie algebra. $\Gamma(G)$ will denote any linear representation of G and $\Gamma(\mathcal{L})$ the representation of the infinitesimal elements of $\Gamma(G)$. Theorem 5 shows that $\Gamma(G) = \exp\Gamma(\mathcal{L})$. A necessary condition for G' to be a subgroup of G is that $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$,(theorem 1) . Thus the problem is : if $\mathcal{L}'$ is a subalg-ebra of $\mathcal{L}$, what are the embedding relations of the Lie groups having Lie algebras isomorphic to $\mathcal{L}'$ in Lie groups having Lie algebras isomorphic to $\mathcal{L}$.

If $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$, then for every linear representation Γ of $\mathcal{L}$ there obviously exists a linear representation Γ' of $\mathcal{L}'$ such that $\Gamma'(\mathcal{L}')$ is a subalgebra of $\Gamma(\mathcal{L})$, (Γ' is a projection of Γ onto $\mathcal{L}'$) . Thus for a representation $\Gamma_G(\mathcal{L})$ which gives a faithful (single-valued) representation $\Gamma(G)$ of G when exponentiated, there exists a linear representation Γ' of $\mathcal{L}'$ such that $\Gamma'(\mathcal{L}') \subset \Gamma_G(\mathcal{L})$. $\Gamma'(\mathcal{L}')$ will give a faithful representation of some group G' when exponentiated (theorem 6) . Thus $\Gamma'(\mathcal{L}') \subset \Gamma_G(\mathcal{L})$ implies $\Gamma'(G') = \exp\Gamma'(\mathcal{L}') \subset \exp\Gamma(\mathcal{L}) = \Gamma_G(G)$, and hence that G' is a subgroup of G .

<u>So if $\mathcal{L}' \subset \mathcal{L}$, then every linear Lie group G whose Lie algebra is isomorphic to $\mathcal{L}$ contain as a subgroup , at least one Lie group G' whose Lie algebra is isomorphic to $\mathcal{L}'$</u> .

Moreover if $\mathcal{L}$ has n non-conjugate semi-simple subalgebras , so does

$\Gamma_G(\mathcal{L})$ and thus G has n non-conjugate semi-simple Lie subgroups . However isomorphic subalgebras of $\mathcal{L}$ do not necessarily correspond to isomorphic subgroups of G , although they do correspond to locally isomorphic subgroups. Also all locally isomorphic groups have the same number of non-conjugate semi-simple Lie subgroups ,corresponding to the same subalgebras of $\mathcal{L}$, though isomorphic subgroups of a group G do not necessarily correspond to isomorphic subgroups of a locally isomorphic group F . Section 4.2.2 illustrates this with the example of the subgroups of groups locally isomorphic to SU(4) , the results being summarised in table 4.3 .

Any representation $\Gamma'(\mathcal{L}')$ of $\mathcal{L}'$, when exponentiated gives a faithful (single-valued) representation $\Gamma'(G')$ of one Lie group G' (unique up to isomorphism) by theorem 6 . Any equivalent representation $\Gamma''(\mathcal{L}') = \underset{\sim}{B}.\Gamma'(\mathcal{L}').\underset{\sim}{B}^{-1}$ will give an equivalent representation $\underset{\sim}{B}.\Gamma'(G').\underset{\sim}{B}^{-1}$ of G' , when exponentiated since

$$\exp(\underset{\sim}{B}.\underset{\sim}{a}.\underset{\sim}{B}^{-1}) = \underset{\sim}{I} + \underset{\sim}{B}.\underset{\sim}{a}.\underset{\sim}{B}^{-1} + \ldots + \frac{(\underset{\sim}{B}.\underset{\sim}{a}.\underset{\sim}{B}^{-1})^r}{r!} + \ldots = \underset{\sim}{B}.\underset{\sim}{I}.\underset{\sim}{B}^{-1} + \ldots \frac{\underset{\sim}{B}.\underset{\sim}{a}^r.\underset{\sim}{B}^{-1}}{r!} \ldots$$

$$= \underset{\sim}{B}(\underset{\sim}{I} + \underset{\sim}{a} + \ldots \frac{\underset{\sim}{a}^r}{r!} + \ldots)\underset{\sim}{B}^{-1} \qquad = \underset{\sim}{B}(\exp \underset{\sim}{a})\,\underset{\sim}{B}^{-1} .$$

Hence ~~So that~~ conjugate embeddings of $\mathcal{L}'$ in $\mathcal{L}$ correspond to conjugate embeddings of G' in G . Thus in order to find all non-conjugate embeddings of G' in G , one must find all non-conjugate embeddings of $\mathcal{L}'$ in $\mathcal{L}$. If $\Gamma_G(G)$ and $\Gamma'(G')$ are faithful representations of G and G' respectively and if $\Gamma'(\mathcal{L}') \subset \Gamma(\mathcal{L})$, then $\Gamma'(G') = \exp \Gamma(\mathcal{L}') \subset \exp \Gamma_G(\mathcal{L}) = \Gamma_G(G)$ and hence G' is a subgroup of G . If G' is a subgroup of G , then it is obvious that the infinitesimal elements of $\Gamma'(G')$ are infinitesimal elements of $\Gamma_G(G)$, where $\Gamma'(G') \subset \Gamma(G)$, both Γ' and Γ being faithful , and thus $\Gamma'(\mathcal{L}')$ is a subalgebra of $\Gamma_G(\mathcal{L})$.

Thus a necessary and sufficient condition for G' to be a subgroup of

G is that there exists a faithful (single-valued) linear representation $\Gamma'(G')$ of G', such that $\Gamma'(\mathcal{L}')$ is a subalgebra of $\Gamma_G(\mathcal{L})$, where $\Gamma_G(G)$ is a faithful representation of G.

$\Gamma(G)$ is a faithful single-valued representation of G if and only if its centre is isomorphic to the centre of G . Any linear semi-simple group G has at least one faithful , single-valued representation , which is completely reduced . Taking one of these (the one of smallest dimension) $\Gamma = \Gamma^1 \oplus \Gamma^2 \oplus \ldots \oplus \Gamma^i \oplus \ldots \Gamma^m$ then any member $\Gamma(z)$ of the centre $\Gamma(Z)$ must be of the form $\Gamma(z) = \Gamma^1(z) \oplus \Gamma^2(z) \oplus \ldots \oplus \Gamma^m(z)$, where $\Gamma^i(z)$ commutes with $\Gamma^i(b)$ for all $b \in G$ ($i = 1,\ldots,m$) and thus Schur's lemma gives $\Gamma^i(z) = \lambda I(n_i)$ (n_i being the dimension of Γ^i) . Since all irreducible finite representations of semi-simple Lie groups are unimodular λ must be an n_ith root of unity and so $\Gamma^i(Z)$ is isomorphic to a cyclic group $I_{n_i'}$ of order $n_i' \leqslant n_i$. Thus $\Gamma(Z)$ is isomorphic to the direct sum of cyclic groups (which is isomorphic to the direct product of cyclic groups) . Thus Z itself must be isomorphic to the direct sum (or direct product) of cyclic groups . If G has a faithful ,single-valued , irreducible representation (of dimension n) , then its centre is the cyclic group $\left\{\omega_r I(n) : \omega_r^n = 1\right\}_{r=1}^{n} \cong I_{n'}$ for some $n' \leqslant n$. The centres of all universal covering groups and universal linear groups can be obtained from [36] and hence the centres of all their their factor groups can be found also . The centre of the direct product of simple groups is the direct product of their centres . Table 4.1 gives the centres and number of locally isomorphic linear groups for all real simple complex Lie algebras . If $\mathcal{L}$ is generated from $\mathcal{L}_c$ by an inner automorphism then its Cartan subalgebra will be the same as that of $\mathcal{L}_c$ and since the central elements are of the form $z = \gamma I = \exp \Gamma(h)$ for some h in $\mathcal{H}$, $\Gamma(G)$ and $\Gamma(G_c)$ will have isomorphic centres . Thus the following sets of groups have isomorphic centres : $\{SU(p,n+1-p): p=0,\ldots[\frac{n}{2}]\}$, $\{SO(2p,2n-2p+1):p=0,\ldots n\}$, $\{Sp(2n), NSp_{2n}^{2p} : p=0,\ldots[\frac{n}{2}]\}$, $\{ND_{2n}, SO(2p,2n-2p): p=0,\ldots[\frac{n}{2}]\}$, $\{CE_6, NE_6^1, NE_6^2\}$, $\{CE_7, NE_7^1, NE_7^2, NE_7^3\}$, $\{CE_8, NE_8^1, NE_8^2\}$, $\{CF_4, NF_4^1, NF_4^2\}$, $\{CG_2, NG_2\}$ and the sets of factor groups of these groups . A similar argument holds for algebras generated by outer automorphisms of the same component . Thus the following sets of groups have isomorphic centres : $\{Q_{\frac{1}{2}(n+1)}, SL(n+1,R)\}$, $\{SO(2p+1,2n-2p-1) : p=0,\ldots[\frac{n}{2}]\}$, $\{NE_6^3, NE_6^4\}$ and the corresponding sets of factor groups .

Table 4.1

Locally isomorphic linear groups having simple complex Lie algebras.

$\tilde{\mathcal{L}}$	$\mathcal{L}$	Centre of $\hat{G}$: $\hat{Z}$	No. of locally isomorphic gps.	Well known locally isomorphic groups
A_n	$SU(p,n+1-p):p=0,.\frac{n+1}{2}$	$I_{(n+1)}$	No. of factors of (n+1)	$SU(p,n+1-p) = \hat{G}$
A_{2n}	$SL(2n+1,R)$	I_1	1	$SL(2n+1,R) = \hat{G}$
A_{2n+1}	Q_n	I_2	2	$Q_n = \hat{G}$
	$SL(2n,R)$	I_2	2	$SL(2n,R) = \hat{G}$
B_n	$SO(2p,2n+1-2p):p=0,.n$	I_2	2	$SO(2p,2n+1-2p) = \frac{\hat{G}}{I_2}$
C_n	$Sp(2n)$	I_2	2	$Sp(2n) = \hat{G}$
	NSp_{2n}^{2p} : $p=1, \dots \frac{n}{2}$	I_2	2	$NSp_{2n}^{2p} = \hat{G}$
	$Sp(n,R)$	I_2	2	$Sp(n,R) = \hat{G}$
D_{2n}	$SO(4n)$	$I_2 \otimes I_2$	3	$SO(4n) = \hat{G}/I_2$
	$SO(4n-2p,2p)$	$I_2 \otimes I_2$	3	$SO(4n-2p,2p) = \hat{G}/I_2$
	ND_{4n}	$I_2 \otimes I_2$	3	$ND_{4n} = \widehat{ND}_{4n}/I_2$
	$SO(4n-2p-1,2p+1)$	I_2	2	$SO(4n-2p-1,2p+1)$ $= \hat{G}/I_2$
D_{2n+1}	$SO(4n+2)$	I_4	3	$SO(4n+2) = \hat{G}/I_2$
	$SO(4n+2-2p,2p)$	I_4	3	$SO(4n+2-2p,2p) = \hat{G}/I_2$
	ND_{4n+2}	I_4	3	$ND_{4n+2} = \hat{G}/I_2$
	$SO(4n+1-2p,2p+1)$	I_2	2	$SO(4n+1-2p,2p+1) = \hat{G}/I_2$
E_6	CE_6, NE_6^1, NE_6^2	I_3	2	
	NE_6^3, NE_6^4	I_1	1	
E_7	All real forms	I_2	2	
E_8, F_4, G_2	All real forms	I_1	1	

A procedure for obtaining all non-conjugate embeddings of a semi-simple Lie group G' in a semi-simple Lie group G is :

1) Find all non-conjugate embeddings of the abstract Lie algebra $\mathcal{L}'$ in the abstract Lie algebra $\mathcal{L}$ (from [10 , 11, 12] and chapters 2 and 3) . Select one $\Gamma^E(\mathcal{L}')$.
2) Find a representation $\Gamma_G(\mathcal{L})$ of $\mathcal{L}$ which provides a faithful(single-valued) representation of G when exponentiated .
3) Find the projection Γ' of Γ_G on $\Gamma^E(\mathcal{L}')$, (i.e. $\Gamma'\Gamma^E(\mathcal{L}') \subset \Gamma_G(\mathcal{L})$). If the centres of $\Gamma'(G')$ and G' are of the same order (i.e. $\Gamma'(G')$ is a faithful (single-valued) representation of G') , then G' is a subgroup of G with representation Γ^E. If the centres of $\Gamma'(G')$ and G' are not isomorphic then G' is not a sub~~algebra~~group of G with any representation conjugate to Γ^E .
4) Repeat 3) with all non-conjugate representations $\Gamma^E(\mathcal{L}')$ for which $\Gamma^E(\mathcal{L}') \subset \Gamma_G(\mathcal{L})$.

The isomorphism theorems (7 and 8) are useful , in that they show that all subgroups of $G = \hat{G}/N$ are of the form G'/N , where G' is a subgroup of $\hat{G}$, and $N \subset G'$. Thus if all the subgroups of $\hat{G}$ are known (including finite groups and the direct product of finite and Lie groups) , then all the subgroups of G are factor groups of the subgroups of $\hat{G}$. Note that a group G'' can be a subgroup of both $\hat{G}$ and G , if $G'' \otimes N$ is a subgroup of $\hat{G}$.

4.2.2 <u>Example</u> : Semi-simple subgroups of groups locally isomorphic to SU(4)

A_3 has A_1 , A_2 , C_2 and $A_1 \oplus A_1$ as subalgebras . Table 4.2 gives the locally isomorphic compact Lie groups with complex Lie algebras A_n,n=1,.,3 C_2 and $A_1 \oplus A_1$, and their centres .

For A_1 , the centre of an irreducible representation Γ_n of $\mathcal{L}'_c$

Table 4.2

Locally isomorphic compact groups for $\check{\mathcal{L}} = A_1, A_2, C_2, A_1 \oplus A_1$.

$\check{\mathcal{L}}$	$\mathcal{L}_c$	$Z(\hat{G})$	Isomorphic groups G	Z(G)
$A_1 \cong B_1 \cong C_1$	SU(2)	I_2	$Sp(2) \cong SU(2) \cong \hat{G}$ $SO(3) \cong \hat{G}/I_2$	I_2 I_1
A_2	SU(3)	I_3	$SU(3) = \hat{G}$ $SU(3)/I_3$	I_3 I_1
$A_3 \cong D_3$	SU(4)	I_4	$SU(4) = \hat{G}$ $SO(6) = \hat{G}/I_2$ $SU(4)/I_4$	I_4 I_2 I_1
$C_2 \cong B_2$	Sp(4)	I_2	$Sp(4) = \hat{G}$ $SO(5) = \hat{G}/I_2$	I_2 I_1
$A_1 \oplus A_1$	$\widehat{SU(2) \otimes SU(2)}$	$I_2 \otimes I_2$	$SU(2) \otimes SU(2) = \hat{G}$ $SO(4) = \hat{G}/I_2$ $SO(3) \otimes SO(3) = \hat{G}/I_2 \otimes I_2$	$I_2 \otimes I_2$ I_2 I_1

N.B. The group $\widehat{SU(2) \otimes SU(2)}$ denotes the group of ordered pairs $\{(a.b) : a, b \subset SU(2)\}$. This group is isomorphic to the group of matrices $SU(2) \oplus SU(2) = \{ a \oplus b : a, b \subset SU(2) \}$, but <u>not</u> to the group of matrices $SU(2) \otimes SU(2) = \{ a \otimes b : a, b \subset SU(2) \}$, which is isomorphic to SO(4) .

consists of elements $\underset{\smile}{c} = \exp\zeta\underset{\sim}{\Gamma}_n(\underset{\sim}{h}_1) = \gamma\underset{\sim}{I}$. Since $\underset{\sim}{\Gamma}_n(\underset{\sim}{h}_1) = \mathrm{diag}(n-1, n-3, \ldots, -n+1)$, $\zeta = m\pi i$ and $\gamma = \pm 1$. If n is odd then one diagonal element is zero, so $\gamma = +1$. Thus the centre is I_1 if Γ only contains odd dimensional representations and I_2 if Γ contains any even dimensional representations. In the former case Γ provides a faithful representation of SO(3) and in the latter case Γ provides a faithful representation of SU(2). For $A_1 \oplus A_1$, a representation of the form $\Gamma_{2n} \otimes \Gamma_{2m}$ has a centre of $\pm \underset{\sim}{I}(2n) \otimes \pm \underset{\sim}{I}(2m) = \pm \underset{\sim}{I}(4nm) \cong I_2, \Gamma_{2n+1} \otimes \Gamma_{2m+1}$ has a centre $\underset{\sim}{I}((2n+1)(2m+1)) \cong I_1, \Gamma_{2n} \otimes \Gamma_{2m+1}$ has a centre $\pm \underset{\sim}{I}(2n(2m+1)) \cong I_2$, and a representation of the form $\Gamma_{2n} \otimes \Gamma_{2m+1} \oplus \Gamma_{2m'+1} \otimes \Gamma_{2n'}$ has a centre of the form $\pm \underset{\sim}{I}(2n(2m+1)) \oplus \pm \underset{\sim}{I}(2n'(2m'+1)) \cong I_2 \otimes I_2$. Thus representations with only odd dimensional representations in their reduction are faithful representations of $SO(3) \otimes SO(3)$ e.g. $\Gamma_3 \otimes \Gamma_1 + \Gamma_1 \otimes \Gamma_3$; those with at least one term $\Gamma_{2n} \otimes \Gamma_{2m+1}$ and one term $\Gamma_{2m'+1} \otimes \Gamma_{2n'}$ are faithful representations of $\widetilde{SU(2) \otimes SU(2)}$ e.g. $\Gamma_2 \otimes \Gamma_1 + \Gamma_1 \otimes \Gamma_2$; and all other representations are faithful representations of SO(4). Since Sp(4) is 4-dimensional, it gives a 4-dimensional representation of $C_2 : \Gamma_4$, and SO(5) gives Γ_5. Similarly SU(4) gives Γ_4 of A_3 and SO(6) gives Γ_6. The centres of SU(4) and $\Gamma_4(A_3)$ are of the same order and so are the centres of $\Gamma_6(A_3), \Gamma_{10}(A_3)$ and SO(6), and the centres of $\Gamma_{15}(A_3)$ and $SU(4)/I_4$. Thus using [33] the semi-simple subgroups of $\Gamma_4(A_3) \cong SU(4)$, $\Gamma_6(A_3) \cong SO(6)$ and $\Gamma_{15}(A_3) \cong SU(4)/I_4$ are found and given in table 4.3.

Table 4.3: Semi-simple Lie subgroups of the compact locally isomorphic groups corresponding to A_3.

$\tilde{\mathcal{L}}'$	Embeddings in $\Gamma_4(A_3)$		Embeddings in $\Gamma_6(A_3)$		Embeddings in $\Gamma_{15}(A_3)$	
(non-conjugate subalgebras of A_3)	Representation	Subgroups of SU(4)	Representation	Subgroups of SO(6)	Representation	Subgroups of $\frac{SU(4)}{I_4}$
A_2	$\Gamma_3 + \Gamma_1$	SU(3)	$\Gamma_3 + \Gamma_3^*$	SU(3)	$\Gamma_8 + \Gamma_3 + \Gamma_3^* + \Gamma_1$	SU(3)
C_2	Γ_4	Sp(4)	$\Gamma_5 + \Gamma_1$	SO(5)	$\Gamma_{10} + \Gamma_5$	SO(5)
$A_1 \oplus A_1$	$\Gamma_2 \otimes \Gamma_2$	SO(4)	$\Gamma_3 \otimes \Gamma_1 + \Gamma_1 \otimes \Gamma_3$	SO(3) ⊗ SO(3)	$\Gamma_3 \otimes \Gamma_1 + \Gamma_3 \otimes \Gamma_1 + \Gamma_3 \otimes \Gamma_3$	SO(3) ⊗ SO(3)
$A_1 \oplus A_1$	$\Gamma_2 \otimes \Gamma_1 + \Gamma_1 \otimes \Gamma_2$	$\widehat{SU(2) \otimes SU(2)}$	$\Gamma_2 \otimes \Gamma_2 + 2\Gamma_1 \otimes \Gamma_1$	SO(4)	$\Gamma_3 \otimes \Gamma_1 + \Gamma_1 \otimes \Gamma_3 + 2\Gamma_2 \otimes \Gamma_2 + \Gamma_1 \otimes \Gamma_1$	SO(4)
A_1	$\Gamma_3 + \Gamma_1$	SO(3)	$\Gamma_3 + \Gamma_3$	SO(3)	$\Gamma_5 + 3\Gamma_3 + \Gamma_1$	SO(3)
A_1	Γ_4	SU(2)	$\Gamma_5 + \Gamma_1$	SO(3)	$\Gamma_7 + \Gamma_5 + \Gamma_3$	SO(3)
A_1	$\Gamma_2 + \Gamma_2$	SU(2)	$\Gamma_3 + 3\Gamma_1$	SO(3)	$5\Gamma_3$	SO(3)
A_1	$\Gamma_2 + 2\Gamma_1$	SU(2)	$\Gamma_2 + \Gamma_2 + \Gamma_1 + \Gamma_1$	SU(2)	$\Gamma_3 + 4\Gamma_2 + 4\Gamma_1$	SU(2)

Appendix 1

DYNKIN DIAGRAMS ,CARTAN MATRICES AND THEIR INVERSES FOR ALL SIMPLE LIE ALGEBRAS .

A.1.1 A_n

Dynkin diagram

Cartan Matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2,-1,0,\ldots\ldots\ldots\ldots 0 \\ -1,\ 2,-1,\ldots\ldots\ldots\ldots 0 \\ 0,-1,\ 2,-1,\ldots\ldots\ldots 0 \\ 0\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 0\ldots\ldots\ldots\ldots -1,\ 2,\ -1 \\ 0\ldots\ldots\ldots\ldots 0,-1\ ,\ 2 \end{pmatrix}$$

$$\underset{\sim}{A}^{-1} = \frac{1}{n+1}\begin{pmatrix} n\ ,\ (n-1)\ ,\ (n-2)\ldots\ldots\ldots,\ r,\ldots\ldots 2,1 \\ n-1,2(n-1)\ ,2(n-2)\ldots\ldots\ldots\ 2r,\ldots\ldots,4,2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ r^2,\ldots\ldots 2r,r \\ r\ ,\ 2r\ ,\ 3r,\ldots\ldots r^2,\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 3\ ,\ 6\ ,\ 9,\ldots\ldots 3r,\ldots\ldots\ldots\ldots\ldots\ldots \\ 2\ ,\ 4\ ,\ 6,\ldots\ldots 2r,\ldots\ldots\ldots\ldots\ldots n-1 \\ 1\ ,\ 2\ ,\ 3\ ,\ldots\ldots r,\ldots\ldots\ldots\ldots n-1,n \end{pmatrix}$$

A.1.2 B_n

Dynkin diagram

Cartan Matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2,-1,0 \ldots\ldots\ldots\ldots\ldots\ldots 0 \\ -1,\ 2,-1, \ldots\ldots\ldots\ldots\ldots 0 \\ 0,\ -1,2,-1, \ldots\ldots\ldots\ldots 0 \\ 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 0 \ldots\ldots\ldots\ldots\ldots -1,\ 2,-1 \\ 0 \ldots\ldots\ldots\ldots\ldots 0,-2,\ 2 \end{pmatrix}$$

$$\underset{\sim}{A}^{-1} = \begin{pmatrix} 2,\ 2,\ 2,\ 2,\ \ldots\ldots\ldots\ldots 2,\ 1 \\ 2,\ 4,\ 4,\ 4, \ldots\ldots\ldots\ldots 4,\ 2 \\ 2,\ 4,\ 6,\ 6, \ldots\ldots\ldots\ldots 6,\ 3 \\ 2,\ 4,\ 6,\ 8, \ldots\ldots\ldots\ldots 8, 4 \\ 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2,\ 4,\ 6,\ 8, \ldots\ldots\ldots 2(n-1), n-1 \\ 2,\ 4,\ 6,\ 8, \ldots\ldots\ldots 2(n-1),\ n \end{pmatrix}$$

A.1.3 C_n

Dynkin diagram

Cartan Matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2,-1,\ 0 \cdots\cdots\cdots\cdots 0 \\ -1,\ 2,-1, \cdots\cdots\cdots\cdots 0 \\ 0,-1,\ 2,-1 \cdots\cdots\cdots\cdots 0 \\ 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\ 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\ 0 \cdots\cdots\cdots\cdots -1,\ 2,-1,0 \\ 0 \cdots\cdots\cdots\cdots\ 0,-1,\ 2,-2 \\ 0, \cdots\cdots\cdots\cdots\cdots\ 0,-1,\ 2 \end{pmatrix}$$

$$\underset{\sim}{A}^{-1} = \begin{pmatrix} 2,\ 2,\ 2,\ 2, \cdots\cdots\cdots\cdots 2 \\ 2,\ 4,\ 4,\ 4, \cdots\cdots\cdots\cdots 4 \\ 2,\ 4,\ 6,\ 6, \cdots\cdots\cdots\cdots 6 \\ 2,\ 4,\ 6,\ 8, \cdots\cdots\cdots\cdots 8 \\ 2, \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 2,\ 4,\ 6,\ 8, \cdots\cdots\cdots 2(n-1), n-1 \\ 1,\ 2,\ 3,\ 4, \cdots\cdots\cdots (n-1),\ n \end{pmatrix}$$

A.1.4 D_n

Dynkin diagram

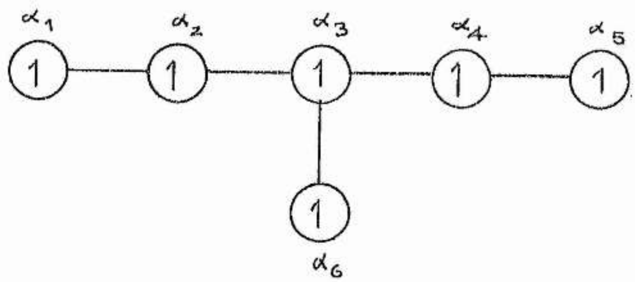
Cartan Matrix

$$
\underset{\sim}{A} = \begin{pmatrix}
2, -1, 0 \ldots\ldots\ldots\ldots\ldots 0 \\
-1, 2, -1, \ldots\ldots\ldots\ldots 0 \\
0, -1, 2, -1, \ldots\ldots\ldots 0 \\
0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\
0 \ldots\ldots\ldots 2, -1, 0, 0 \\
0 \ldots\ldots\ldots -1, 2, -1, -1 \\
0 \ldots\ldots\ldots 0, -1, 2, 0 \\
0 \ldots\ldots\ldots 0, -1, 0, 2
\end{pmatrix}
$$

$$
\underset{\sim}{A}^{-1} = \frac{1}{4}\begin{pmatrix}
4, 4, 4, 4, \ldots\ldots\ldots 4, 2, 2 \\
4, 8, 8, 8, \ldots\ldots\ldots 8, 4, 4 \\
4, 8, 12, 12, \ldots\ldots\ldots 12, 6, 6 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
4, 8, 12, 16, \ldots\ldots\ldots 2(n-2), 2(n-2) \\
2, 4, 6, 8, \ldots\ldots\ldots n, (n-2) \\
2, 4, 6, 8, \ldots\ldots\ldots (n-2), n
\end{pmatrix}
$$

A.1.5 E_6

Dynkin diagram

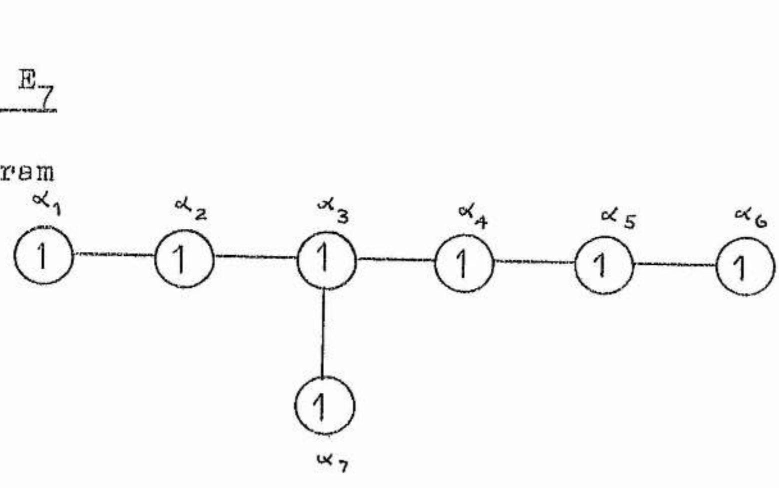


Cartan Matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2,&-1,&0,&0,&0,&0\\ -1,&2,&-1,&0,&0,&0\\ 0,&-1,&2,&-1,&0,&-1\\ 0,&0,&-1,&2,&-1,&0\\ 0,&0,&0,&-1,&2,&0\\ 0,&0,&-1,&0,&0,&2 \end{pmatrix} \qquad \underset{\sim}{A}^{-1} = \frac{1}{3}\begin{pmatrix} 4,&5,&6,&4,&2,&3\\ 5,&10,&12,&8,&4,&6\\ 6,&12,&18,&12,&6,&9\\ 4,&8,&12,&10,&5,&6\\ 2,&4,&6,&5,&4,&3\\ 3,&6,&9,&6,&3,&6 \end{pmatrix}$$

A.1.6 E_7

Dykin diagram

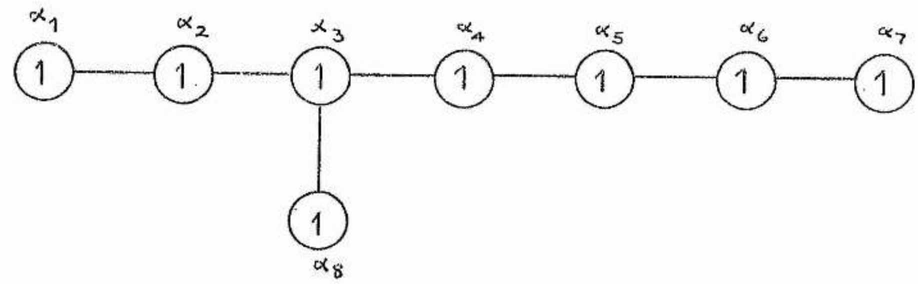


Certan Matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2,&-1,&0,&0,&0,&0,&0\\ -1,&2,&-1,&0,&0,&0,&0\\ 0,&-1,&2,&-1,&0,&0,&-1\\ 0,&0,&-1,&2,&-1,&0,&0\\ 0,&0,&0,&-1,&2,&-1,&0\\ 0,&0,&0,&0,&-1,&2,&0\\ 0,&0,&-1,&0,&0,&0,&2 \end{pmatrix} \qquad \underset{\sim}{A}^{-1} = \begin{pmatrix} 2,&3,&4,&3,&2,&1,&2\\ 3,&6,&8,&6,&4,&2,&4\\ 4,&8,&12,&9,&6,&3,&6\\ 3,&6,&9,&7,&5,&2,&4\\ 2,&4,&6,&5,&4,&2,&3\\ 1,&2,&3,&2,&2,&1,&1\\ 2,&4,&6,&4,&3,&1,&3 \end{pmatrix}$$

A.1.7 E_8

Dynkin diagram



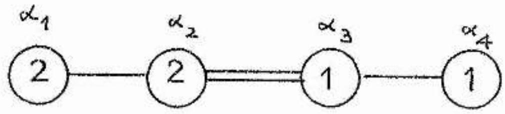
Cartan matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2, & -1, & 0, & 0, & 0, & 0, & 0, & 0 \\ -1, & 2, & -1, & 0, & 0, & 0, & 0, & 0 \\ 0, & -1, & 2, & -1, & 0, & 0, & 0, & -1 \\ 0, & 0, & -1, & 2, & -1, & 0, & 0, & 0 \\ 0, & 0, & 0, & -1, & 2, & -1, & 0, & 0 \\ 0, & 0, & 0, & 0, & -1, & 2, & -1, & 0 \\ 0, & 0, & 0, & 0, & 0, & -1, & 2, & 0 \\ 0, & 0, & -1, & 0, & 0, & 0, & 0, & 2 \end{pmatrix}$$

$$\underset{\sim}{A}^{-1} = \begin{pmatrix} 4, & 7, & 10, & 8, & 6, & 4, & 2, & 5 \\ 7, & 14, & 20, & 16, & 12, & 8, & 4, & 10 \\ 10, & 20, & 30, & 24, & 18, & 12, & 6, & 15 \\ 8, & 16, & 24, & 20, & 15, & 10, & 5, & 12 \\ 6, & 12, & 18, & 15, & 12, & 8, & 4, & 9 \\ 4, & 8, & 12, & 10, & 8, & 6, & 3, & 6 \\ 2, & 4, & 6, & 5, & 4, & 3, & 2, & 3 \\ 5, & 10, & 15, & 12, & 9, & 6, & 3, & 8 \end{pmatrix}$$

A.1.8 F_4

Dynkin diagram

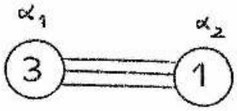


Cartan matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2, & -1, & 0, & 0 \\ -1, & 2, & -1, & 0 \\ 0, & -2, & 2, & -1 \\ 0, & 0, & -1, & 2 \end{pmatrix} \qquad \underset{\sim}{A}^{-1} = \begin{pmatrix} 2, & 3, & 2, & 1 \\ 3, & 6, & 4, & 2 \\ 4, & 8, & 6, & 3 \\ 2, & 4, & 3, & 2 \end{pmatrix}$$

A.1.9 G_2

Dynkin diagram



Cartan matrix

$$\underset{\sim}{A} = \begin{pmatrix} 2, & -1 \\ -3, & 2 \end{pmatrix} \qquad \underset{\sim}{A}^{-1} = \begin{pmatrix} 2, & 1 \\ 3, & 2 \end{pmatrix}$$

Appendix 2

EXPLICIT MATRIX FORMS FOR THE CLASSICAL SIMPLE LIE ALGEBRAS (IN CANONICAL FORM)

A.2.1 A_ℓ

$$\left.\begin{aligned} \underset{\sim}{h}_j &= \underset{\sim}{I}_{j+1,j+1} - \underset{\sim}{I}_{j,j} \\ \underset{\sim}{e}_j &= I_{j,j+1} \quad , \quad \underset{\sim}{f}_j = -\tilde{\underset{\sim}{e}}_j \end{aligned}\right\} \qquad j = 1,\ldots\ldots,\ell$$

A.2.2 B_ℓ

$$\left.\begin{aligned} \underset{\sim}{h}_j &= -\underset{\sim}{I}_{j+1,j+1} + \underset{\sim}{I}_{j+2,j+2} + \underset{\sim}{I}_{j+\ +1,j+\ +1} - \underset{\sim}{I}_{j+\ +2,j+\ +2} \\ \underset{\sim}{e}_j &= \underset{\sim}{I}_{j+1,j+2} - \underset{\sim}{I}_{j+\ +2,j+\ +1} \end{aligned}\right\}$$

$$j = 1,\ldots,\ell-1$$

$$\underset{\sim}{h}_\ell = 2(-\underset{\sim}{I}_{\ell+1,\ell+1} + \underset{\sim}{I}_{2\ell+1,2\ell+1})$$

$$\underset{\sim}{e}_\ell = \sqrt{2}(\underset{\sim}{I}_{1,2\ell+1} - \underset{\sim}{I}_{\ell+1,1})$$

$$\underset{\sim}{f}_j = -\tilde{\underset{\sim}{e}}_j \qquad j = 1,\ldots\ldots,\ell$$

A.2.3 C_ℓ

$$\left.\begin{aligned} \underset{\sim}{h}_j &= -\underset{\sim}{I}_{j,j} + \underset{\sim}{I}_{j+1,j+1} + \underset{\sim}{I}_{j+\ell,j+\ell} - \underset{\sim}{I}_{j+\ell+1,j+\ell+1} \\ \underset{\sim}{e}_j &= \underset{\sim}{I}_{j.j+1} - \underset{\sim}{I}_{j+\ell+1,j+\ell} \end{aligned}\right\}$$

$$j = 1,\ldots\ldots,\ell-1$$

$$\underset{\sim}{h}_\ell = -\underset{\sim}{I}_{\ell,\ell} + I_{2\ell,2\ell}$$

$$\underset{\sim}{e}_\ell = \underset{\sim}{I}_{\ell,2\ell}$$

$$\underset{\sim}{f}_j = -\tilde{\underset{\sim}{e}}_j \qquad j = 1,\ldots\ldots\ldots,\ell$$

A.2.4 D_ℓ

$$\left.\begin{aligned} \underset{\sim}{h}_j &= -\underset{\sim}{I}_{j,j} + \underset{\sim}{I}_{j+1,j+1} + \underset{\sim}{I}_{j+\ell,j+\ell} - \underset{\sim}{I}_{j+\ell+1,j+\ell+1} \\ \underset{\sim}{e}_j &= \underset{\sim}{I}_{j,j+1} - \underset{\sim}{I}_{j+\ell+1,j+\ell} \end{aligned}\right\}$$

$$j=1,\ldots\ldots,\ell-1$$

$$\underset{\sim}{h}_\ell = -\underset{\sim}{I}_{\ell-1,\ell-1} - \underset{\sim}{I}_{\ell,\ell} + \underset{\sim}{I}_{2\ell-1,2\ell-1} + \underset{\sim}{I}_{2\ell,2\ell}$$

$$\underset{\sim}{e}_\ell = \underset{\sim}{I}_{\ell-1,2\ell} - \underset{\sim}{I}_{\ell,2\ell-1}$$

$$\underset{\sim}{f}_j = -\tilde{\underset{\sim}{e}}_j \qquad j=1,\ldots\ldots\ldots,\ell$$

APPENDIX 3

THEOREMS

Theorem 1 If G' is a subalgebra of G , then its Lie algebra $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$, the Lie algebra of G .

Proof

$\mathcal{L}$ is the algebra of the infinitesimal elements of G , so that every element $\underset{\sim}{g} \in G$, in a neighbourhood of the identity $\underset{\sim}{I}$ can be expressed

$\underset{\sim}{g} = \underset{\sim}{I} + t_i \underset{\sim}{a}_i$, where the $\underset{\sim}{a}_i (i = 1,..n)$ are the generators of $\mathcal{L}$,which has dimension n , Similarly all elements $\underset{\sim}{g}' \epsilon G'$ in a neighbourhood of the identity can be expressed as

$\underset{\sim}{g}' = \underset{\sim}{I} + t_i' \underset{\sim}{a}_i'$. But since all $\underset{\sim}{g}' \epsilon G'$ also belong to G , for any generator $\underset{\sim}{a}_j'$ of $\mathcal{L}'$ there exists a $\underset{\sim}{g}' \epsilon G'$ and G such that

$\underset{\sim}{g}' = \underset{\sim}{I} + \gamma' \underset{\sim}{a}_j' = \underset{\sim}{I} + \gamma_{ij} \underset{\sim}{a}_i$ for some set γ_{ij} ($i=1,..n$; $j=1,..n'$)

Thus $\underset{\sim}{a}_j' = \frac{1}{\gamma} \gamma_{ij} \underset{\sim}{a}_i$ which belongs to $\mathcal{L}$. Thus all the generators of $\mathcal{L}'$ belong to $\mathcal{L}$, and $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$.

Theorem 2 All automorphisms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$ ($\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ being simple) are of type(i) if $\tilde{\mathcal{L}}_1 \not\cong \tilde{\mathcal{L}}_2$ and of type (i) or type (ii) if $\tilde{\mathcal{L}}_1 \cong \tilde{\mathcal{L}}_2$.

Proof

Gantmacher [19 and 20] has proved that every automorphism A of a semi-simple Lie algebra $\mathcal{L}$ induces a rotation in the root space V of $\mathcal{L}$, and every rotation τ of the root space can be completed to an automorphism of $\mathcal{L}$. Thus $(A-\tau)\, \underset{\sim}{h} = \underset{\sim}{0}$ for all $\underset{\sim}{h}$ in $\mathcal{H}$, and $A\underset{\sim}{e}_\alpha = \chi_\alpha \underset{\sim}{e}_{\alpha^*}$ where $\chi_\alpha = \pm 1$ and $\tau\alpha = \alpha^*$, He has also shown that an inner automorphism corresponds to a reflection σ about a plane perpendicular to a root , $\mathfrak{S}$ being the group of all such reflections . An outer automorphism corresponds to a 'particular'rotation , (one which induces a rotation of the simple roots in the Dynkin diagram)

The group of automorphisms can be decomposed into the sum of connected components :

$$\mathcal{U} = \mathcal{U}_o + \mathcal{U}_1 + \ldots\ldots\ldots + \mathcal{U}_i \ldots\ldots$$

and each component $\mathcal{U}_i$ corresponds to a <u>chief</u> particular rotation τ_i , τ_o being the identity . The automorphisms of $\mathcal{U}_i$ then induce the rotations of $\tau_i\ \mathfrak{S}$, all of which can only be completed to give automorphisms which belong to $\mathcal{U}_i$. Thus in examining the automorphisms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$, it is sufficient to examine the rotations of the root space $V = V_1 + V_2$, where V_1 and V_2 are the orthogonal root spaces of $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ respectively .

There is a class of rotations , induced by automorphisms of type (i) , which can be expressed as a product $\tau_1 . \tau_2$ of a rotation τ_1 in V_1 and a rotation τ_2 in V_2 . All elements of $\mathfrak{S}$ must be of this form, since they correspond to inner automorphisms , which are of type (i) : $T(g_1 \otimes g_2) = \hat{T}(g_1).\hat{T}(g_2)$ for all $g_1 \in G_1$ and $g_2 \in G_2$. So it is only particular rotations which may not be of this form .

Suppose there exists a rotation τ which does not correspond to an automorphism of type (i) . Then $\tau V_1 \neq V_1$ and $\tau V_2 \neq V_2$. Thus at least one root of V_1 , $\alpha^{(1)}_j$, is mapped into V_2 and at least one root of V_2 , $\alpha^{(2)}_j$, is mapped into V_1 . The roots of V_1 and V_2 can then be arranged :

$$\alpha^{(1)}_j \in V_1 \quad , \quad \tau\,\alpha^{(1)}_j \in \begin{cases} V_2 & \text{for } j = 1,\ldots q \,. \\ V_1 & \text{for } j = q+1,\ldots,k_1 . \end{cases}$$

$$\alpha^{(2)}_j \in V_2 \quad , \quad \tau\,\alpha^{(2)}_j \in \begin{cases} V_1 & \text{for } j = 1,\ldots\ldots r \,. \\ V_2 & \text{for } j = r+1,\ldots,k_2 \,. \end{cases}$$

where $\tilde{\mathcal{L}}_i$ has k_i roots (i=1,2) . Since τ is an automorphism of V $(\alpha,\beta) = (\tau\alpha, \tau\beta)$ for all $\alpha,\beta \in V$. Thus :

$(\alpha^{(1)}_i, \alpha^{(2)}_j) = (\tau\,\alpha^{(1)}_i, \tau\,\alpha^{(2)}_j) = 0 \quad i= 1,\ldots k_1 \ , \ j= 1,\ldots k_2 \ ,$ and

$(\tau\alpha_i^{(1)}, \tau\alpha_j^{(1)}) = (\alpha_i^{(1)}, \alpha_j^{(1)}) = 0$, $i = 1,\ldots q$, $j = q+1,\ldots k_1$.

This implies that V_1 consists of two orthogonal components unless $q = 0$ or k_1 . This would contradict the assumption that $\tilde{\mathcal{L}}_1$ was simple . Thus $q = 0$ or k_1 and similarly $r = 0$ or k_2 . q and r both zero corresponds to automorphisms of type (i) and since V is finite $V_1+V_2 \xrightarrow{\tau} V_1$ or V_2 is not an automorphism , it is not possible for q to be zero and r non-zero or vice versa . Thus the only possiblity is $q=k_1$ and $r=k_2$. So $\tau V_1 \equiv V_2$ and $\tau V_2 \equiv V_1$. Since τ is an automorphism this implies that V_1 and V_2 are isomorphic , and hence if τ is completed to A , $A\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2$ and $A\tilde{\mathcal{L}}_2 = \tilde{\mathcal{L}}_1$, which is an automorphism of type (ii). This also implies that $\tilde{\mathcal{L}}_1 \cong \tilde{\mathcal{L}}_2$. Thus if $\tilde{\mathcal{L}}_1 \cong \tilde{\mathcal{L}}_2$ then the automorphisms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$ are either of type (i) or type (ii) and if $\tilde{\mathcal{L}}_1 \ncong \tilde{\mathcal{L}}_2$ then the automorphisms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$ are of type (i) .

<u>Corollary</u> The automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ are of type (i) if $\mathcal{L}_{1c} \ncong \mathcal{L}_{2c}$, and of type (i) or type (ii) if $\mathcal{L}_{1c} \cong \mathcal{L}_{2c}$.

<u>Proof</u>

All automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ can be extended to give automorphisms of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$. Thus all automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ can be obtained from those of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$. $\mathcal{L}_{1c} \cong \mathcal{L}_{2c} \Longleftrightarrow \tilde{\mathcal{L}}_1 \cong \tilde{\mathcal{L}}_2$. Since an automorphism of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$ will be of the same type when it is reduced to $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$, the automorphisms of $\mathcal{L}_{1c} \oplus \mathcal{L}_{2c}$ will be of the same type as those of $\tilde{\mathcal{L}}_1 \oplus \tilde{\mathcal{L}}_2$.

<u>Theorem 3</u> All representations Γ of a simple Lie algebra $\mathcal{L}$ are either faithful or trivial .

<u>Proof</u>

Since $\mathcal{L}$ is simply-connected Γ can not be multi-valued .

If Γ is not faithful then there exists a set S of $\mathcal{L}$ defined by :

$S = \{ a : \Gamma(a) = b \}$ for some b, which contains more than one element. If a_1 and a_2 are two distinct elements of S then $\Gamma(a_1 - a_2) = 0$. Hence a set of at least two elements can be defined by

$K = \{ k : \Gamma(k) = 0 \} = \{ 0, (a_1 - a_2), \ldots \}$. Since Γ is a representation $\Gamma([k,a]) = [\Gamma(k), \Gamma(a)] = [0, \Gamma(a)] = 0$. So K is an ideal and $\mathcal{L}$ is not simple. This contradicts the initial assumption that $\mathcal{L}$ is simple and so either K is trivial (Γ faithful) or $K = \mathcal{L}$, (Γ trivial). Thus Γ is either trivial or faithful.

Theorem 4

a) A faithful linear representation of a semi-simple Lie group G provides a faithful linear representation of its Lie algebra $\mathcal{L}$.
b) A faithful linear representation of a semi-simple Lie algebra $\mathcal{L}$ provides a linear representation of any group G having $\mathcal{L}$ as its Lie algebra (not necessarily faithful or single-valued).
N.B. faithful also implies single-valued here.

Proof

a) If Γ is faithful and linear then $\Gamma(G) \subset GL(n,C)$, (n being the dimension of Γ) and there is a one to one correspondence between the elements of G and those of $\Gamma(G)$, (i.e. between g and $\Gamma(g)$, $g \in G$).
If $\Gamma(\mathcal{L})$ is defined as the algebra generated from the elements $\Gamma(a_i)$:

$$\Gamma(a_i) = \lim_{t \to 0} \left\{ \frac{\Gamma(\exp t\, a_i) - I}{t} \right\} \qquad (A.1)$$

where the a_i are the generators of $\mathcal{L}$, (i.e. $\Gamma(\mathcal{L})$ is the Lie algebra of $\Gamma(G)$). Then since $GL(n,C)$ is closed $\Gamma(a_i) \in GL(n,C)$, so $\Gamma(\mathcal{L})$ is linear. In a neighbourhood of the identity there is a one to one correspondence between g and $\Gamma(g)$ and thus between $I + t_i a_i$ and $I + t_i \Gamma(a_i)$ and between a_i and $\Gamma(a_i)$ for all the generators a_i of $\mathcal{L}$. Thus $\Gamma(\mathcal{L})$ is a faithful representation of $\mathcal{L}$.

b) If $\Gamma(\mathcal{L})$ is a faithful linear representation of $\mathcal{L}$, then $\Gamma(\mathcal{L}) \in GL(n,C)$ and so $\exp \Gamma(\mathcal{L}) \in GL(n,C)$. Thus by defining $\Gamma(G)$ by

$$\Gamma(g) = \exp \Gamma(a) \tag{A.2}$$

where $g = \exp a$ and so

$$\Gamma(\exp a) = \exp \Gamma(a) \tag{A.3}$$

Then

$$\Gamma(\exp a . \exp b) = \Gamma(\exp(a+b+ \tfrac{1}{2}[a,b] + \tfrac{1}{12}[a,[a,b]] + \tfrac{1}{12}[b,[b,a]] \ldots \tag{A.4}$$

by the Campbell-Baker-Hausdorf formula and

$$\Gamma(\exp a).\Gamma(\exp b) = \exp\Gamma(a).\exp\Gamma(b) = \exp(\Gamma(a)+\Gamma(b)+\tfrac{1}{2}\ \Gamma(a),\Gamma(b) \ldots \tag{A.5}$$

Since Γ is a representation of $\mathcal{L}$, $\Gamma(a+b) = \Gamma(a)+\Gamma(b)$ and $\Gamma([a,b]) = [\Gamma(a),\Gamma(b)]$. Substituting into (A.5) and substituting from (A.3) in (A.4) , both equations can easily be shown to be equal . Thus $\Gamma(g_1 . g_2) = \Gamma(g_1).\Gamma(g_2)$ and so Γ is a representation of G .

<u>Theorem 5</u> If $\Gamma(\mathcal{L})$ is the Lie algebra generated by the infinitesimal elements of $\Gamma(G)$, then $\exp \Gamma(\mathcal{L}) = \Gamma(G)$.

<u>Proof</u>

Since $\Gamma(\mathcal{L})$ is the Lie algebra of $\Gamma(G)$ as defined in (A.1) , in a neighbourhood of the identity

$$\Gamma(g) = \Gamma(I + t_i a_i \ldots) \tag{A.6}$$

and

$$\exp\Gamma(t_i a_i) = I + \Gamma(t_i a_i) + \ldots \tag{A.7}$$

since Γ is a representation of $\mathcal{L}$, (theorem 4) (A.6) becomes equal to (A.7) and so $\Gamma(G) = \exp \Gamma(\mathcal{L})$, where $\Gamma(\mathcal{L})$ is the algebra of the infinitesimal elements of $\Gamma(G)$.

<u>Theorem 6</u> Any faithful linear representation Γ of a Lie algebra provides a faithful (single-valued) representation of one Lie group G (unique up to isomorphism) ,when exporentiated .

Proof

By theorem 4(b) $\exp\Gamma(\mathcal{L})$ is a single-valued representation of the universal covering group $\tilde{G}$ of all locally isomorphic Lie groups with Lie algebras isomorphic to $\mathcal{L}$ (all representations of $\tilde{G}$ are single-valued) . Thus $\exp\Gamma(\mathcal{L}) = \Gamma(\tilde{G})$, which is a Lie group since it is continuous, closed, has an identity element and all elements have an inverse . Thus there is an abstract Lie group G (unique up to isomorphism) for which $\Gamma(G) = \exp\Gamma(\mathcal{L})$ is a faithful representation.

Isomorphism Theorems

The proofs of these theorems will not be given as they appear in most standard group theory texts .

Theorem 7: If H is a normal subgroup of G and G' is a subgroup of G , then $H \cap G'$ is a normal subgroup og G' and G'/H G' is isomorphic to H.G'/H .

Theorem 8 :

a) If $H \subset G' \subset G$ and H is a normal subgroup of G , then it is normal in G' and G'/H is a subgroup of G/H . Conversely every subgroup of G/H is of the form G'/H for some G' between G and H .

b) If H is a normal subgroup of G' and G, and G' is a normal subgroup of G , then G'/H is a normal subgroup of G/H and

$$\frac{G/H}{G'/H} = \frac{G}{G'}$$

Conversely every normal subgroup of G/H is of the form G'/H for some G' , which is a normal subgroup of G and of which H is a normal subgroup .

REFERENCES

1 Barut ,A.O. and Raczka ,R . ,Proc.Roy.Soc. A287(1965),519 .

2 Behrends ,R.E. ,Landoritz ,L.F. and Tunkelang ,B. , Phys.Rev. 142 (1966),1092.

3 Behrends ,R.E. and Sirlin , A.,Phys.Rev. 142 (1966),1095 .

4 Behrends ,R.E. ,Phys.Rev. 142 (1966), 1101 .

5 Boerner , H., Representations of Groups ,(North Holland,Amsterdam,1963)

6 Bohm, D.,Flato,M.,Sternheimer, D.,Vigier, J.P. , Nuovo Cimento 38 , (1965),1941 .

7 Cartan,M.E.,Ann.L'Ecole Norm.Sup. 3-ième série, 31 (1914),263 .

8 Cornwell,J.F., Reports Math.Phys. 2 (1971), 153 .

9 Cornwell,J.F., Reports Math.Phys. 2 (1971), 229 .

10 Cornwell,J.F., Reports Math.Phys. 2 (1971), 239 .

11 Cornwell,J.F., Reports Math.Phys. 2 (1971), 289 .

12 Cornwell,J.F., Reports Math.Phys. 3 (1972), 95 .

13 Dynkin, E.B.,Trudy Mosk.Math.Obsc. 1(1952), 39 .(translated in Amer.Math.Soc.Transl.Series 2 ,6 (1965), 245)

14 Dynkin, E.B.,Mat.Sbornik N.S. 30 (1952) ,349 (translated in Amer.Math.Soc.Transl.Series 2, 6 (1965), 111)

15 Flato,M. and Sternheimer, D.,J.Math.Phys. 7 (1966), 1932 .

16 Flato,M. and Sternheimer, D.,Com.Math.Phys. 12 (1969),296.

17 Flato,M. and Snettman , (preprint 1972) Regge Lattice and Lie algebra representations .

18 Gell-Mann,M. and Ne'eman,Y., The Eightfold Way (Benjamin,N.York,1964)

19 Gantmacher,F.,Rec.Math.(Mat.Sbornik) N.S. 5 (47) (1939), 101 .

20 Gantmacher,F.,Rec.Math.(Mat.Sbornik) N.S. 5 (47) (1939), 217 .

21 Halbwachs,F.,Nuovo Cimento 49A (1967), 517 .

22 Hamermesh,M., Group Theory and its applications to Physical problems, (Addison-Wesley, Massachusetts, 1962).

23 Hegerfeldt, G.C. and Hennig, J.,Forschr.Phys. 16,(1968), 491 .

24 Jacobson, N., Lie Algebras , (Interscience,New York,1962)

25 Jost, R.,Helv.Phys.Acta 39 ,(1966), 369 .

26 Mal'cev, A.I.,Rec.Math.(Mat Sbornik) N.S. 16 (58) (1945),163.

27 Mal'cev, A.I.,Rec.Math.(Mat Sbornik) N.S. 19 (61) (1946),523.

28 Mal'cev, A.I.,Izv.Akad.Nauk.SSSR.Ser.Mat. 8 (1944) , 143 . (translated in Amer.Math.Soc.Transl.,Series 1,9 (1953), 173 .

29 Mehta, M.L.,J.Math.Phys. 7 (1966) , 1824 .

30 Mehta,M.L., and Srivastava, P.K.,J.Math.Phys. 7 (1966),1833.

31 Patera, J.,J.Math.Phys. 11 (1970), 3027 .

32 Patera,J. ,J.Math.Phys. 12 (1971), 384 .

33 Patera,J. and Sankoff,D., Branching rules for representations of Simple Lie algebras, Centre de Recherches Mathematiques,Univer. de Montreal (preprint CRM-167, Feb.1972) .

34 O'Raifeartaigh, L.,Phys.Rev. 139 (1965),1052 .

35 Segal, I.,J.Functional Analysis 1 (1966), 1.

36 Sirota, A.I. and Solondovnikov,A.S.,Rus,Math.Surveys 18 (1963)3,85.

37 Tait. W., Int.J.Theor.Phys. 6 (1972), 453 .

38 Tinkham,M.,Group Theory and Quantum Mechanics,(McGraw-Hill,N.York 1946).

39 Weyl,H., Classical Groups (Princeton Univ.Press,Princeton,1939).

40 Barut, A.O. and Duru, I.A.,Proc.Roy.Soc. A 333,(1973),217 .

41 Barut, A.O. and Monsma.W.B.,Phys.Rev. DJ (1972), 2327 .